Topological entropy and periodic points for $\mathbb{Z}^d$ actions on compact abelian groups with the Descending Chain Condition

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This little book is simply packed
With scientific information,
And I can vouch for every fact
And almost every calculation;
But if you think they represent
A life of entomologism,
A life of rash experiment
With every kind of organism,
You musn't - mine were different media;
I owe my facts to my Encyclopædia.

A. P. Herbert.

For my mother and father

Contents

Chapter I: Introduction.

1. The Descending Chain Condition p.3
2. Locally Compact Abelian Groups p.9
3. Algebraic Background p.11

Chapter II: Topological Entropy.

1. Topological Entropy for Group Actions p. 13
2. Periodic Points for Group Actions p. 20
3. Growth rate of Periodic Points and Topological Entropy p. 28
4. Examples and Remarks p. 41
5. Cohomology and Periodic Points p. 45

Chapter III: Algebraic Entropy.

1. Topological Entropy and Algebraic Entropy p. 49

Chapter IV: Zeta Functions.

1. Group Actions and ζ-functions p. 58
2. Poles and Product Formulae p. 62
3. Poles of the ζ-function and the Mahler measure p. 69

Appendix A: *Automorphisms of solenoids and p-adic entropy.* Doug Lind and Tom Ward. p. 72

Appendix B: The Entropy Formula. p. 81

Appendix C: Calculations of Periodic Points. p. 93

Bibliography. p. 98

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Declaration.

Most of the results in this thesis have benefited from conversations with Klaus Schmidt, my supervisor. The central section II.3 is joint work with Klaus Schmidt. For expansive systems, most of the results of section II.3 will appear in Lind, Schmidt and Ward [1].

The paper comprising Appendix A is joint work with Doug Lind. The results and methods of this paper have been previously used by me in my Master of Science dissertation at the University of Warwick. This paper has appeared in Ergodic Theory and Dynamical Systems.

The material in Appendix B is due to Doug Lind and Klaus Schmidt, and will appear in Lind, Schmidt and Ward [1]. This paper contains material without which this thesis could not have been written.

The material in Chapter III is formally original in that it relates to higher dimensional actions but is a trivial rewriting of the paper by Justin Peters [1].

The remaining material is original except where specific details are given.

Summary.

This thesis deals with the ergodic theory of actions of $\mathbb{Z}^d$ on compact abelian groups satisfying a regularity condition: the descending chain condition.

Using results of Klaus Schmidt, Bruce Kitchens and Doug Lind, we show that the descending chain condition guarantees that the global topological entropy of such an action coincides with the growth rate of periodic points whenever this exists (§II.3). This links the global entropy to the dynamics of such systems and allows the global entropy to be computed for some important examples (§II.4).

The algebraic entropy of a $\mathbb{Z}^d$ action on a discrete abelian group is defined. Following Justin Peters, we show that this entropy coincides with the topological entropy of the adjoint action on the compact dual group (§III.1).

Two possible zeta functions for actions are introduced, and their poles are found for expansive systems with the descending chain condition (§IV.2). We compare these functions with the one-dimensional case of a single automorphism, where the least real pole of the zeta function lies at exp(-entropy). We relate an instance of convergence at a pole with the Mahler measure of a polynomial described as a limit of one dimensional Mahler measures (§IV.3).

Appendix A is a paper written with Doug Lind in which the basic one-dimensional entropy formula (Yuzvinskii's formula) is computed using adelic methods.

Appendix B reproduces for the sake of completeness the higher-dimensional entropy formula due to Doug Lind and Klaus Schmidt.

Appendix C contains some examples of calculations of numbers of periodic points for simple higher-dimensional systems.

I: Introduction

In this thesis we study some of the topological ergodic theory of actions of $\mathbb{Z}^d$ on compact abelian groups that satisfy the Descending Chain Condition, defined below. This condition restricts the structure of the compact group and allows a complete algebraic description of such actions. This is an extension of Theorem 19 in Lawton [1], which shows that if α is an automorphism of a compact connected abelian group X and the system (X, α) has the property that the dual group X^* is finitely generated under the dual automorphism α^* then X is a solenoid. The condition that X^* be finitely generated under the dual automorphism is equivalent to the Descending Chain Condition. It is also shown (Theorem 14) that if X is any compact group admitting an expansive automorphism, then X must be finite dimensional. The stronger result that any compact connected group admitting an expansive family (semigroup) of endomorphisms must be abelian and metrizable was shown by Ping-Fum Lam (Theorem 3.2 in Lam [1]). As pointed out there, it is essential to assume that the group is connected since the identity map on any finite group is expansive. Bruce Kitchens and Klaus Schmidt [1] have shown that any expansive action of a finitely generated abelian group on a compact group satisfies the DCC. Further, it is shown (see Theorem 7.3 in Kitchens and Schmidt [1]), that if X is connected and $\mathbb{Z}^d$ acts expansively on X then X must be abelian.

The principal result in this thesis concerns the relationship between the global topological entropy of such actions and the growth rate of their periodic points. This shows directly that positive global topological entropy reflects complicated dynamical properties and provides a method for computing this important invariant in certain cases. The contents are arranged as follows.

In Section I.1 we define the basic notions: that of a $\mathbb{Z}^d$ action and the Descending Chain Condition. We state a structure theorem due to Bruce Kitchens and Klaus Schmidt. This allows for actions with the Descending Chain Condition to be studied using the methods of commutative algebra and algebraic geometry as well as abelian harmonic analysis. We (implicitly) introduce the category of d-dimensional actions on compact abelian groups with the Descending Chain Condition. This category, whose morphisms are the group homomorphisms commuting with the actions, is isomorphic by the structure theorem to the category *Mod(d)* of finitely generated $\mathbb{Z}[u_1, \ldots, u_d, u_1^{-1}, \ldots, u_d^{-1}]$ modules, with morphisms given by $\mathbb{Z}[u_1, \ldots, u_d, u_1^{-1}, \ldots, u_d^{-1}]$ module homomorphisms. We also introduce the subcategory of systems determined by ideals, *Id(d)*. Section I.2 very briefly introduces the theory of abelian harmonic analysis on compact groups which is used throughout. Section I.3 describes enough commutative algebra to allow a finitely-generated $\mathbb{Z}[u_1, \ldots, u_d, u_1^{-1}, \ldots, u_d^{-1}]$ module to be decomposed into ideal systems using a prime filtration or into primary modules using a primary decomposition of zero. Modules that are not finitely generated correspond to actions that do not satisfy the Descending Chain Condition.

In Section II.1 we define the global topological entropy of a $\mathbb{Z}^d$ action on a compact metric space by homeomorphisms. This involves the notion of Følner sequences in amenable groups. We mention some of the more general results on Følner sequences in amenable groups. Following the theory of topological entropy for homeomorphisms due to Rufus Bowen we give a formulation of the entropy locally (II.1.4). In II.1.6 we show how to express a general system as a skew product of simpler ideal systems. Finally, we quote the formula for the entropy of $\mathbb{Z}^d$ actions due to Doug Lind and Klaus Schmidt which relates the entropy to a logarithmic Mahler measure.

Periodic points for actions are introduced in Section II.2. We also introduce a canonical subgroup of the group of points of given period that allows the theory to be developed even in the presence of infinitely many fixed points for the action.

The main theorem is shown in Section II.3. We show that the entropy of a d-dimensional action can be obtained as a limit of the entropies of certain (d-1)-dimensional actions. This allows for an inductive proof of the theorem II.3.5 that the growth rate of periodic points along suitable periods equals the entropy.

In Section II.4 some examples of entropy values are given, and we show directly for some expansive systems that the growth rate of periodic points coincides with the entropy.

Section II.5 interprets the presence of periodic points that cannot be lifted through a filtration in terms of a non-trivial cohomology group for (a subsystem of) the action.

Section III.1. extends a result due to Justin Peters. We show, using methods identical to Peters, that the topological entropy of a $\mathbb{Z}^d$ action on a compact abelian group is equal to the algebraic entropy of the dual action on the discrete abelian dual group. The algebraic entropy can be computed by combinatorial methods. This does not depend on a regularity condition and merely illustrates the duality of Fourier Analysis.

Section IV.1 defines two possible zeta functions for expansive actions, and computes some simple examples. In Section IV.2 we find where the poles of the zeta functions lie (IV.2.2 and IV.2.3). In particular, we show that the poles of the rectangular zeta function reflect all the possible entropies of the subsystems of lower dimension. Further (IV.2.4), we show that the essential pole (which corresponds to the d-dimensional or global entropy) lies in the closure of the other poles. We also check that the zeta functions satisfy Euler-Lagrange product formulae. Section IV.3 is a brief sketch of how some of the above results fit into the theory of Mahler measures.

Appendix A is a paper jointly written by Doug Lind and the author. This paper gives an accessible derivation of Yuzvinskii's basic formula for the entropy of an automorphism of a solenoid, by covering a solenoid with a finite dimensional vector space over the adele group $\mathbb{Q}_A$. This is analogous to covering a toral automorphism with an automorphism of a finite dimensional vector space over $\mathbb{R}$.

Appendix B comprises a statement of the addition formula for the entropy of higher dimensional actions, and a proof of the entropy formula for actions determined by ideals. The proof is due to Doug Lind and Klaus Schmidt and is reproduced from Lind, Schmidt and Ward [1].

Appendix C is a selection of calculations to find the number of points of small periods for a few simple examples of higher dimensional actions.

§1: The Descending Chain Condition

We now proceed to define these notions and provide references for the main results.

Definition I.1.1: An action[1] of a finitely generated abelian group Γ on a compact abelian group X, described as a pair (X, Γ) is said to satisfy the *descending*

[1] All actions are assumed to be actions by continuous automorphisms, that is they are given by a homomorphism $\Gamma \longrightarrow \mathrm{Aut}(X)$ where $\mathrm{Aut}(X)$ is the group of continuous automorphisms of X. Such actions are convenient for ergodic theory because they are automatically measure-preserving actions on the probability space (X, Borel sets, Haar measure). This was first pointed out by Halmos [1] in 1943. There are discontinuous algebraic automorphisms of tori whose definition is nonconstructive. They turn out to be nonmeasurable and are therefore not of direct interest in ergodic theory.

chain condition if for every chain $X \supset V_1 \supset V_2 \supset \ldots \supset V_n \supset \ldots$ of closed Γ-invariant subgroups there exists an integer $N \geq 1$ with $V_n = V_N$ for all $n \geq N$.

In Kitchens and Schmidt [1], this is considered for any compact group (Definition 3.1) and a structure theorem is then shown for such systems (Theorem 3.2). We are only interested in the abelian case where there is the following (Theorem 11.8).

Theorem I.1.2: Let $\Gamma = \mathbb{Z}^d$ act on the compact abelian group X by the homomorphism $\eta : \mathbb{Z}^d \longrightarrow \mathrm{Aut}(X)$, with the property that the pair (X, Γ) has the descending chain condition. Then there exists a finitely generated $\mathbf{R}$-module $\mathbf{M}$, where $\mathbf{R}$ is the ring $\mathbb{Z}[u_1, \ldots, u_d, u_1^{-1}, \ldots, u_d^{-1}]$, a continuous isomorphism $\psi : \mathbf{M}^* \longrightarrow X$, where $\mathbf{M}^*$ is the dual of $\mathbf{M}$, such that $\psi\alpha_n(y) = \eta(n)\psi(y)$ for all $y \in \mathbf{M}^*$ and $\mathbf{n} \in \mathbb{Z}^d$. Here α_n is the dual automorphism to multiplication by $u_1^{n_1} \ldots u_d^{n_d}$ on $\mathbf{M}$.

We will say that the system is in $\mathcal{Mod}(d)$ to describe this situation, and if $\mathbf{M}^*$ is connected we will say the system is in $\mathcal{Mod}_c(d)$. By Theorem 24.25 of Hewitt and Ross [1], $\mathbf{M}^*$ is connected if and only if $\mathbf{M}$ is torsion free.

The papers Kitchens and Schmidt [1] and more particularly Schmidt [1] go on to relate properties of the module $\mathbf{M}$ to dynamical properties of the system (X, Γ). There are characterisations of ergodicity (11.5 in Kitchens and Schmidt [1]), mixing (11.7 in Kitchens and Schmidt [1]), finiteness of the number of periodic points of any given period (3.6 in Schmidt [1]) and expansiveness (3.8 in Schmidt [1]).

We will normally consider the case where $\mathbf{M}$ is of the form $\mathbf{R}/\mathbf{p}$, where $\mathbf{p}$ is a prime ideal in $\mathbf{R}$ generated by a finite set of polynomials $f_1, \ldots, f_l$. We will say that a given polynomial $f = \sum f_n u^n$ is in standard form if its support, defined as $\{\mathbf{n} \in \mathbb{Z}^d \mid f_n \neq 0\}$, meets each axis in $\mathbb{Z}^d$ and every coordinate of every point in the support is non-negative. For polynomials in standard form there is a well-defined notion of degree in each of the variables. Notice that any non-zero polynomial is associated (in the ring of Laurent polynomials) to a unique polynomial in standard form. Write $V(\mathbf{p})$ for the set of zeros of $\mathbf{p}$ in $(\mathbb{C}^{\times})^d$. Say that a set $S \in \mathbb{C}$ has no unit roots if $S \cap \mathcal{U} = \emptyset$ where $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ and $\mathcal{U}_n$ is given by:-

$$\mathcal{U}_n = \{ (z_1, \ldots, z_d) \mid z_1^n = \ldots = z_d^n = 1 \}$$

The descending chain condition on $X_{R/p}$ is equivalent to the Noetherian condition on the ring **R** (see VII.4.9 of Hungerford [1]). When the ideal is principal, $p = \langle f \rangle$, we will use the suffix f instead of **R/p**. In either of these situations $\mathbf{R} = \mathbf{R}(d)$ will be the ring of Laurent polynomials in d variables over the integers, $X = X_{R/p}$ or X_f will be the dual of **R/p** or $\mathbf{R}/\langle f \rangle$. The action of $\mathbb{Z}^d$ on **R** will be defined by β (sometimes written $\beta_{R/p}$), where:-

$$\beta_n(g) = u_1^{n_1} \ldots u_d^{n_d} g .$$

The action on $\mathbf{R}^*$ is defined by α, sometimes written $\alpha_{R/p}$, the adjoint or dual action to β. That is, $\alpha_n(x)(g) = x(\beta_n(g))$ for all $x \in X_{R/p}$. A subgroup **p** is an ideal of **R** if and only if **p** is an β-invariant subset, so we automatically induce a well-defined action of β on the quotient **R/p** and hence on the dual of the quotient $X_{R/p}$. We will call the class of such systems *Id(d)*. The subclass of connected systems will be denoted $Id_c(d)$. By Theorem 24.25 of Hewitt and Ross [1] as above $X_{R/p}$ is in $Id_c(d)$ if and only if **R/p** is torsion free. This occurs if and only if the generators of **p** do not have a non-trivial constant common factor up to association in **R**.

The class *Id(d)* is a very convenient one for calculations but the following example shows that it is not a natural one dynamically since it is not closed under direct products.

Example I.1.3: The system $Y = X_{R/\langle 2\rangle} \times X_{R/\langle 2\rangle}$ with action given by $\alpha = \alpha_{R/\langle 2\rangle} \times \alpha_{R/\langle 2\rangle}$ is not contained in the class of ideal systems $Id(d) = \{(X_{R/p}, \mathbb{Z}^d) \mid \mathbf{p} \text{ is an ideal in } \mathbf{R}\}$. This shows that *Id(d)* is not closed under the taking of cartesian products. We note that (Y, α) is topologically conjugate to $(X_{R/\langle 4\rangle}, \alpha_{R/\langle 4\rangle})$ in *Id(d)* but they are of course not algebraically conjugate.

This example is considered in Remark 4.4 of Schmidt [1] to show that neither an Arov[2] nor an Adler-Palais[3] type result holds for all systems in *Mod(d)*. In the same paper it is however shown that topological conjugacy does imply algebraic conjugacy for systems in $\mathcal{Mod}_c(d)$ – the subclass of connected systems (see Theorem 4.2 in Schmidt [1]).

A further, and for our purposes more serious, problem is that the $\mathbb{Z}^{d-1}$ action defined on the points of period n in a chosen direction for a given element of *Id(d)* is not in general an element of *Id(d-1)*. However, we can canonically construct an element of *Id(d-1)* that has the same entropy as such a system.

Lemma I.1.4: The classes $\mathcal{Mod}(d)$ and $\mathcal{Mod}_c(d)$ are each closed under the taking of cartesian products.

Proof: This is clear, and can be seen directly as follows. Given **R**–modules **N** and **L**, $\mathbf{N} \times \mathbf{L}$ also lies in $\mathcal{Mod}(d)$ since $\mathbf{N} \times \mathbf{L}$ is a finitely generated **R**–module. Alternatively, notice that the product of two actions of two finitely generated abelian groups on compact abelian groups with the descending chain condition is a system of the same type and hence by Theorem I.1.2 lies in $\mathcal{Mod}(d)$.

The systems in question are connected if and only if the duals are torsion free so $\mathcal{Mod}_c(d)$ is closed under taking products since for any groups G and H, $G \times H$ is torsion free if and only if G and H are. □

We will say that systems of the above kind are d–dimensional. Thus, to say that a system is one–dimensional does not mean that the compact group X has topological dimension one but that the acting group is $\mathbb{Z}$.

Notice that if $\mathbf{p} = \{0\}$ then we have the full d–dimensional shift on the lattice of circles $\mathbb{T}^{\mathbb{Z}^d}$. If $\mathbf{p} = \{m\}$ where m is a non–zero integer we have the full d–dimensional shift on $|m|$ symbols. Replacing $\mathbb{Z}$ by $\mathbb{Z}/m\mathbb{Z}$ for some m produces the full shift on m symbols if $\mathbf{p} = \{0\}$ and in general produces a finite characteristic family of dynamical systems each of which has finite global entropy and is totally

[2]Arov [1] shows that if (X, T) and (Y, S) are $\mathbb{Z}$-actions with the descending chain condition and X and Y are solenoids then the systems are topologically conjugate (there exists a homeomorphism $\emptyset:X \longrightarrow Y$ with $\emptyset T = S\emptyset$) if and only if they are algebraically conjugate (there exists a continuous group isomorphism $\theta:X \longrightarrow Y$ with $\theta T = S\theta$). This is discussed in Kitchens and Schmidt [1] where it is shown that the functor $H_1(\,,\mathbb{T})$ fixes solenoids since they are projective limits of tori. Given $\emptyset$ we can take $\theta = H_1(\emptyset,\mathbb{T})$.

[3]Adler and Palais [1] show that if S and T are ergodic automorphisms of the torus $\mathbb{T}^n$ and there is a topological conjugacy $\emptyset$ between them then $\emptyset$ is an automorphism of $\mathbb{T}^n$ composed with a rotation by a fixed point of T.

disconnected[4]. These finite characteristic systems exhibit complex mixing properties (see Schmidt [2] for general results about higher dimensional mixing on compact groups). We mention one example due to Francois Ledrappier [1]: the system $(X_{<2, 1+x+y>}, \mathbb{Z}^2)$ is mixing but not 2-mixing. For more examples and results on this finite characteristic case see Kitchens and Schmidt [1].

Example I.1.5: As an illustration of the **R**-module formalism, consider the following one-dimensional example of metric conjugacy without topological conjugacy, due to R. F. Williams [1]. Let G be the two-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$, and let β_A and β_B be the automorphisms of G defined by the unimodular matrices:

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}.$$

Then the matrices A and B have the same characteristic polynomial $\chi_A(x) = \chi_B(x) = x^2 - 4x - 1$, and hence the same entropy. Thus the dynamical systems (G, β_A, μ_G) and (G, β_B, μ_G) are metrically isomorphic by section 5 of Adler and Weiss [1]. However, it can be shown[5] that A and B are not conjugate elements

[4]For a non-trivial ideal these examples have zero global entropy. This shows that the computations of entropy in Lind, Schmidt and Ward [1] do not provide any solution to the interesting problem of computing the global entropy for general higher-dimensional Markov shifts with finite alphabet. In order for an element of $\mathcal{Id}(d)$ to have a finite alphabet, we must assume that the ideal defining the system contains a constant. If the ideal contains any further elements, we have no entropy. This situation can be illustrated conveniently by writing {d-dimensional Markov shifts with positive entropy} $\cap$ $\mathcal{Id}(d)$ = {d-dimensional Bernoulli shifts on s symbols with measure $(1/s, \ldots, 1/s)$}.

[5]If X is an integer matrix with AX = XB then it can be shown that $\det X \in 2\mathbb{Z}$. This is done in Section 11, (3) of Williams [1]. A similar example is considered in Adler and Weiss [1]. For a more systematic way of studying these problems we may apply Olga Taussky's simplification of the beautiful theorem of Latimer and Macduffee. Latimer and Macduffee [1] have shown that there is a one-to-one correspondence between conjugacy classes of integer matrices and elements of the ideal class group of the ring extension of $\mathbb{Z}$ defined by adjoining a root of each irreducible factor of the characteristic equation. For the case of an irreducible characteristic polynomial, the method has been considerably simplified by Taussky [1]. Following her procedure, from the eigenvalue $\lambda = (2+\sqrt{5})$ we obtain the ring $\mathcal{R} = \mathbb{Z}[2+\sqrt{5}]$. Now the eigenvector corresponding to λ for A is $(2+\sqrt{5}, 1)$ which corresponds to the principal ideal $\mathcal{I} = (2+\sqrt{5}, 1) = (1) = \mathcal{R}$ of $\mathcal{R}$. The eigenvector corresponding to λ for B is $\mathcal{J} = (1+\sqrt{5}, 2)$. The fact that $\mathcal{I}$ is principal and $\mathcal{J}$ is non-principal means they define distinct elements of the ideal class group of $\mathcal{R}$ and so A and B are not conjugate. Furthermore, because $\mathcal{I}$ defines the identity element of the ideal class group, A is conjugate to the

of $GL(2, \mathbb{Z})$ and so the automorphisms cannot be topologically conjugate ($\equiv$ algebraically conjugate by Arov [1]). In the module framework, we describe the two systems by defining $\mathbf{R}(1)$ modules $\mathbf{M}$ and $\mathbf{L}$ with $(X_M, \alpha_M) \cong (G, \beta_A)$ and $(X_L, \alpha_L) \cong (G, \beta_B)$. This is done by setting:

$$\mathbf{M} = \frac{\mathbf{R}^2}{\begin{bmatrix} 4-x & 1 \\ 1 & -x \end{bmatrix}\mathbf{R}^2} \; ; \; \mathbf{L} = \frac{\mathbf{R}^2}{\begin{bmatrix} 3-x & 2 \\ 2 & 1-x \end{bmatrix}\mathbf{R}^2}.$$

The above remark can now be expressed by noting that the additive groups of $\mathbf{M}$ and $\mathbf{L}$ are isomorphic (to the dual of G), but that $\mathbf{M}$ and $\mathbf{L}$ are not isomorphic as $\mathbf{R}(1)$ modules. This may be seen by computing the additive group of the factor modules $\mathbf{M}' = \mathbf{M}/(1-x)\mathbf{M}$ and $\mathbf{L}' = \mathbf{L}/(1-x)\mathbf{L}$. We find that $\mathbf{L}' \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ while $\mathbf{M}' \cong \mathbb{Z}/4\mathbb{Z}$. This shows the modules are not isomorphic. The groups $\mathbb{Z}/4\mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ are isomorphic to the groups of fixed points in (G, β_A) and (G, β_B) respectively.

This example is discussed in Kitchens and Schmidt [1] (Examples 6.6 (4)) with A replaced by the matrix:

$$C = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}.$$

This is however conjugate to A:

$$C = \begin{bmatrix} 4 & 5 \\ 1 & 1 \end{bmatrix}^{-1} A \begin{bmatrix} 4 & 5 \\ 1 & 1 \end{bmatrix}.$$

Example I.1.6: A two dimensional example is given by the following matrices. Let

companion matrix of the polynomial χ_A. This analysis can be carried much further. For instance, since B is symmetric, Theorem 1 in Taussky [2] shows that $\mathcal{J}$ is self inverse in the ideal class group. Theorem 4 in Taussky [3] gives criteria to guarantee that a class of matrices corresponding to a self-inverse element of the ideal class group contains a symmetric matrix, as in this example.

(X_A, α_A) and (X_B, α_B) be the $\mathbb{Z}^2$ action defined by the modules $\mathbf{R}^2/A\mathbf{R}^2$ and $\mathbf{R}^2/B\mathbf{R}^2$ respectively, where $\mathbf{R} = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ and

$$A = \begin{bmatrix} x+2y & y \\ 4 & 3 \end{bmatrix}; \; B = \begin{bmatrix} x+2y & 4y \\ 1 & 3 \end{bmatrix}.$$

Section II.3 will show that these systems have equal entropies, but they are not topologically conjugate. This may be seen by computing the ideal classes corresponding to A and B in the ideal class group of $\mathbf{R}[\lambda]/\langle\lambda^2 - \lambda(3 + x + 2y) + (3x + 2y)\rangle$. However, it is not known if systems of this kind are (higher-dimensional) Bernoulli, so there is no immediate reason to suspect that (X_A, α_A) and (X_B, α_B) are metrically conjugate. The system (X_B, α_B) is conjugate to the ideal system defined by the module $\mathbf{R}/\langle 3x + 2y\rangle$.

Example I.1.7: The system (X_M, α_M), where $\mathbf{M}$ is the module $\mathbf{R}/\langle 3 + x + y\rangle$, is expansive by Schmidt [1]. Write $\alpha = \alpha_M$. It is clear that each α_n has infinite (one dimensional) entropy for each $\mathbf{n} \neq 0$, and so α_n is not an expansive automorphism of X_M. This means that (X_M, α_M) is a properly expansive transformation group of a compact connected group, answering in the affirmative the question raised at the end of Eisenberg [1].

§2: Locally Compact Abelian Groups

We deal always with topological groups and establish the following conventions. All groups will be locally compact metrizable topological groups, by homomorphism we will always mean continuous homomorphism, we will use the symbol $\cong$ for topological group isomorphism and if a group G is abelian we will write G^* for the Pontryagin dual[6] of G (see for instance Section 4.2 of Reiter [1]). We will use

[6] G^* is the group of all continuous homomorphisms from G into the circle group $\mathbb{T}$. When equipped with the topology of uniform convergence on compact sets, G^* becomes a locally compact abelian group in its own right, and the celebrated Pontryagin-van Kampen duality theorem asserts that the dual G^{**} of this locally compact abelian group is isomorphic in a canonical manner to the original group G. For a proof of this, see Reiter [1] or Hewitt and Ross [1].

measure to mean Haar measure normalised to make $\mu(X) = 1$ if X is a compact group and $\mu(0_G) = 1$ if G is a discrete group. All metrics on groups are assumed to be translation invariant. If H is a closed subgroup of G then the annihilator of H in G^*, written $H^\perp$, is the set of all characters in G^* that are identically equal to 1 on H; $H^\perp$ is then a closed subgroup of G^* (see 23.24 in Hewitt and Ross [1]). Also, for closed subgroups H and H' of G, there are (by Theorem 23.25 in Hewitt and Ross [1]) the relations:- $(G/H)^* \cong H^\perp$, $G^*/H^\perp \cong H^*$, $H^\perp + H'^\perp \cong (G/(H \cap H'))^*$ and $H^{\perp\perp} \cong H$. We have the following elementary lemma.

Lemma I.2.1: If C is a cyclic group then C^* is isomorphic to C. If G is a finite abelian group then $|G^*| = |G|$.

Proof: Let C be generated by x, with $|x| = |C| = n$. Then it is clear that if $w \in C^*$ then w is determined by $w(x)$ and this quantity must be an n^{th} root of unity. So C^* is generated by w, where $w(x) = \exp(2\pi i/n)$, and C^* is therefore cyclic of order n.

If G is compact, then G^* forms a complete orthonormal sequence in $L^2(G)$. Hence, $\dim_{\mathbb{C}} L^2(G) = |G^*|$. But by Plancherel's Theorem (See IV.4.5 in Reiter [1]) we have a bijection $L^2(G) \longrightarrow L^2(G^*)$, and so by duality, $|G| = |G^*|$ if either of these quantities is finite. This could also be seen by expressing G as a product of cyclic groups but the maps behind the argument we have used are much more canonical. □

The dual of a homomorphism $\theta: G \longrightarrow H$ is a homomorphism $\theta^*: H^* \longrightarrow G^*$ defined by setting $\theta^*(\chi)(g) = \chi(\theta(g))$ for all $\chi \in H^*, g \in G$. The map θ is injective (surjective) if and only if the dual θ^* is surjective (injective).

In general, discrete groups will be written additively with identity 0 and compact groups will be written multiplicatively with identity 1. By a *solenoid* we will mean a (finite dimensional) compact, connected, abelian group[7], or equivalently a group

[7]These groups are called solenoids by analogy with electrical solenoids because a one-dimensional solenoidal group is compact but contains a copy of the real line wrapped inside it as a dense subgroup: the inclusion $\mathbb{Q} \hookrightarrow \mathbb{R}$ has dense image so the dual of $\mathbb{R}$ (which is a copy of $\mathbb{R}$) lies densely in the dual of $\mathbb{Q}$ (the solenoid) by restricting characters on $\mathbb{R}$ to the subgroup $\mathbb{Q}$. A solenoid of topological dimension n is a quotient of $(\mathbb{Q}_A)^n$ where $\mathbb{Q}_A$ is the adele group (group of *valuation vectors*) of the rationals. This is proved in Hewitt and Ross [1] but is stated somewhat differently there. A direct proof using the language of adeles may be found in Weil [1], §IV.2, where it is shown that the dual of an algebraic number field k is isomorphic to the quotient group k_A/k where k is embedded diagonally as $x \longrightarrow (x, x, \ldots) \in k_A$.

whose dual is discrete, torsion free and abelian. We can express this by saying that a group G is a solenoid if and only if its dual group G^* is a subgroup of $\mathbb{Q}^d$ for some d. For references to results in topological groups, see Hewitt and Ross [1].

§3: Algebraic Background

In this section we assemble the facts about rings, ideals and modules that we will be using. All the material can be found in Chapter VI of Lang [1]. The algebra assembled here can also be found in exactly this context in Section 2 of Schmidt [1].

Let $\mathbf{R}$ be the ring of Laurent polynomials in d variables as in Section 1. In all that follows, let $\mathbf{M}$ be a finitely generated $\mathbf{R}$-module. Then $\mathbf{M}$ is automatically a quotient of a free $\mathbf{R}$-module. For every $r \in \mathbf{R}$, write $r_{\mathbf{M}}:\mathbf{M} \longrightarrow \mathbf{M}$ for the map $r_{\mathbf{M}}(a) = r.a$ for all $a \in \mathbf{M}$. Let $\mathrm{Ann}(a) = \{r \in \mathbf{R} \mid r_{\mathbf{M}}(a) = 0\}$ denote the $\mathbf{R}$-annihilator of $a \in \mathbf{M}$. A prime ideal $\mathbf{p} \subset \mathbf{R}$ is said to be *associated with* $\mathbf{M}$ if there is an element $a \in \mathbf{M}$ for which $\mathbf{p} = \mathrm{Ann}(a)$. The module $\mathbf{M}$ is *associated with* $\mathbf{p}$ if $\mathbf{p}$ is the only prime ideal associated with $\mathbf{M}$. We then have (Lang [1], Corollary VI.4.11): $\mathbf{M}$ is associated with $\mathbf{p}$ if and only if:-

$$\mathbf{p} = \{ r \in \mathbf{R} \mid r_{\mathbf{M}} \text{ is not injective} \} = \{ r \in \mathbf{R} \mid r_{\mathbf{M}} \text{ is nilpotent} \}$$

If $\mathbf{M}$ is associated with $\mathbf{p}$ and $\mathbf{N} \subset \mathbf{M}$ is a non-zero submodule then $\mathbf{N}$ is also associated with $\mathbf{p}$.

A submodule $\mathbf{N} \subset \mathbf{M}$ is $\mathbf{p}$-*primary* (where $\mathbf{p}$ is a prime ideal of $\mathbf{R}$) if $\mathbf{M}/\mathbf{N}$ is associated with $\mathbf{p}$. By Theorem VI.5.3 of Lang [1], there exist $\mathbf{p}_i$-primary submodules $\mathbf{W}_i$ of $\mathbf{M}$ for $i = 1, \ldots, m$ with the following:-

$$\mathbf{W}_1 \cap \mathbf{W}_2 \cap \ldots \cap \mathbf{W}_m = \{0\},$$

for any S a proper subset of $\{1, \ldots, m\}$, $\bigcap_{i \in S} \mathbf{W}_i \neq \{0\}$,

and the primes $\mathbf{p}_i$ corresponding to the submodules $\mathbf{W}_i$ are all distinct.

Such a family of submodules is called a *reduced primary decomposition* of $0 \in \mathbf{M}$, and $\{\mathbf{p}_i\}$ is called the set of *associated primes* of $\mathbf{M}$. The set of associated primes is independent of the decomposition and furthermore (see Lang [1], VI.5.5):-

$$\{ r \in \mathbf{R} \mid r_M : \mathbf{M} \longrightarrow \mathbf{M} \text{ is not injective} \} = \mathbf{p}_1 \cup \ldots \cup \mathbf{p}_m$$

The existence of a finite reduced primary decomposition and the finiteness of the set of associated primes is shown in Proposition 7G of Matsumura [1].

We quote the following (Corollary 2.2 in Schmidt [1]).

Lemma I.3.1: Let $\{\mathbf{p}_1, \ldots, \mathbf{p}_m\}$ be the set of primes associated with **M**. Then there exist submodules $\mathbf{M} = \mathbf{N}_1 \supset \mathbf{N}_2 \supset \ldots \supset \mathbf{N}_s = \{0\}$ such that, for every $i \in \{1, 2, \ldots, s-1\}$, $\mathbf{N}_i/\mathbf{N}_{i+1}$ is isomorphic to $\mathbf{R}/\mathbf{q}_i$ for some prime ideal $\mathbf{q}_i \subset \mathbf{R}$ with $\mathbf{q}_i \supset \mathbf{p}_j$ for some $j \in \{1, 2, \ldots, m\}$.

The chain of submodules given in the above lemma will be called a *prime filtration* of the module **M**. It allows certain properties of systems in $\mathcal{Id}(d)$ to be extended to systems in $\mathcal{Mod}(d)$. The associated primes $\{\mathbf{p}_1, \ldots, \mathbf{p}_m\}$ determine many dynamical properties of the system $(X_\mathbf{M}, \alpha_\mathbf{M})$.

II: Topological Entropy

§1: Topological Entropy for Group Actions

In this section we define the global topological entropy of an action of $\mathbb{Z}^d$ on a compact metric abelian group by continuous automorphisms (or indeed on any compact metric space by homeomorphisms) by means of spanning and separating sets. A convenient framework for a general definition is to use Følner sequences in the acting group.

The theory of metric entropy for an abelian group of transformations of a Lebesgue space is described in Conze [1]. He shows (Theorem 2.3) that if S and T are two automorphisms of the Lebesgue space $(X, \mathcal{A}, \mu)$ with $ST = TS$, and either S or T has finite metric entropy, then the $\mathbb{Z}^2$ action defined by the group of automorphisms generated by S and T has zero global entropy.

Definition II.1.1: A Følner sequence in a locally compact group G with Haar measure μ is a sequence $A_1 \subset A_2 \subset A_3 \subset \ldots \subset G$ of Borel sets with $\mu(A_n) \in (0,\infty)$ for all n and with the property that given $\varepsilon > 0$ and B a compact neighbourhood of $0 \in G$, there exists N such that $n \geq N$ implies:-

$$\frac{\mu(A_n \Delta (B+A_n))}{\mu(A_n)} < \varepsilon$$

Remarks II.1.2:

(i) G is *amenable*[1] if and only if such sequences exist. This is proved in Emerson and Greenleaf [1] and in fact Ollagnier [1] calls such a sequence a (left) ameaning filter. For the original definition see Følner [1].

(ii) If G is abelian then it is amenable by the Kakutani-Markov fixed point theorem (see 4.1.2 of Zimmer [1]). The groups we consider are abelian and discrete so finite

[1]A group G is *amenable* if any action of G by affine continuous one-to-one mappings on a compact and convex subset of a locally convex Hausdorff topological vector space has at least one invariant point. The importance of this property is that it implies that a continuous action of G on a compact metric spaces carries a G-invariant probability measure.

Haar measure can be replaced by finite in the above definition.

(iii) Example. The sets $E_k = E(k) = \{-k, \ldots, 0, \ldots, k\}$ define a Følner sequence in $\mathbb{Z}$.

(iv) For $\mathbb{Z}^d$ we can sometimes consider a rectangular Følner sequence (cf. example after lemma 3.6.4 in Greenleaf [1]), that is to say one in which each A_n takes the form:-

$$A_n = \prod_{i=1}^{d} [0, m_i(n)] \cap \mathbb{Z}^d.$$

This is because for any Følner sequence $B_1 \subset B_2 \subset \ldots$ we can find a sequence of the above form with B_n contained in a translate of A_n for each n, so along a sequence of the above rectangular form we eventually contain translates of the elements of any other sequence.

We now turn to the definition of topological entropy. The s, r notation is originally due to Bowen [2] and follows Chapter 7 of Walters [1]. The definition could be extended to actions by commuting uniformly continuous homeomorphisms on non-compact metric spaces by using Bowens definition for the $d = 1$ case. This is described in Walters [1].

Definition II.1.3: The (global or joint) topological entropy of an action T of $\mathbb{Z}^d$ on a compact metric space (X, d) is defined as follows. Choose a Følner sequence F_n in $\mathbb{Z}^d$. A set $F \subset X$ is said to be (n, ε)-separated if $x, y \in F$ with $x \neq y$ implies that there is an $\mathbf{n} \in F_n$ for which $T_{\mathbf{n}}$ ε-separates x and y, that is, has $d(T_{\mathbf{n}}x, T_{\mathbf{n}}y) \geq \varepsilon$.

Let $s_n(\varepsilon)$ be the maximum cardinality of an (n, ε)-separated set. Define the topological entropy (*a priori* along the sequence F_n) to be:-

$$h_{top}(X) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{|F_n|} \log s_n(\varepsilon)$$

Define similarly with spanning sets, writing $r_n(\varepsilon)$ for the minimum cardinality of an (n, ε)-spanning set. The two quantities coincide as usual since $r_n(\varepsilon) \le s_n(\varepsilon) \le r_n(\varepsilon/2)$, and the second limit (in ε) exists since $h_{top}(X, \varepsilon)$ increases as ε decreases to zero.

Notice that the quantity h is independent of the Følner sequence chosen. For a proof of this in a more general setting and a more detailed discussion of this definition, see section 5.1 of Ollagnier [1]. Notice by Remark II.1.2 *(i)* that the topological entropy of any amenable group action can be defined in this way, replacing $|F_n|$ with $\mu(F_n)$ if μ is Haar measure on the acting group.

From now on we apply Definition II.1.3 to our systems, where X is a compact abelian group of the form X_M as defined in Chapter 1, and T is the action α_M. We will write h_M for the global entropy of α_M on X_M.

Notice that these groups are compact metric spaces with respect to a translation invariant metric. This follows from the Descending Chain Condition or can be seen directly. An example of a translation invariant metric on a group of the form $X_{R/q}$ for some ideal $\mathbf{q} \subset \mathbf{R}$ may be obtained by restricting d, a translation invariant metric on $\mathbb{T}^{\mathbb{Z}^d}$, to the closed subgroup X. Define d as follows. For $x, y \in \mathbb{T}$ let

$$|x - y| = \min\{|a - b + n|_{\mathbb{R}} \mid n \in \mathbb{Z}\}.$$

Then set:

$$d(x,y) = \sum_{n \in \mathbb{Z}^d} 2^{-\max\{|n_1|,\ldots,|n_d|\}} \times |x(\mathbf{n}) - y(\mathbf{n})|\,.$$

The metric d induces a translation invariant metric on closed subgroups of $\mathbb{T}^{\mathbb{Z}^d}$.

We now go through the method of Bowen [2] to use Haar measure to measure volume expansion and compute entropy. This gives a local formula for the entropy.

Definition II.1.4: Define, analogously to $k(\mu, \alpha)$ in Bowen [2], the quantity $\mathrm{Bow}(\alpha_M)$ as follows. Firstly, choose a Følner sequence F_n in $\mathbb{Z}^d$ and let:-

$$D_n(x, \varepsilon) = \bigcap_{\alpha \in F_n} \alpha^{-1} B_\varepsilon(\alpha(x)).$$

If μ is normalized Haar measure on X, notice that:-

(i) $\mu(K) \leq 1$ for all measurable $K \subset X$.

(ii) $\mu(K) > 0$ for any compact K with interior.

(iii) $\mu(D_n(y, \varepsilon)) = \mu(D_n(x, \varepsilon))$ for all $x, y \in X$ by translation invariance of Haar measure.

This means that μ is α_n- *homogeneous* in the sense of Definition 6 of Bowen [2] for each $\mathbf{n}$ in $\mathbb{Z}^d$. Define:-

$$\mathrm{Bow}(\alpha) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \{ \frac{-1}{|F_n|} \log \mu(D_n(x, \varepsilon))\}.$$

This is independent of x by *(iii)* and is independent of the Følner sequence by Lemma II.1.5 below and the comments after Definition II.1.3.

Lemma II.1.5: $\mathrm{Bow}(\alpha_M) = h(\alpha_M)$.

Proof: The proof of this lemma is identical to that of the one dimensional case, which is Proposition 7 in Bowen [2]. Let $F \subset X$ be (n, ε)-separated. Then the sets $D_n(x, \varepsilon/2)$ are disjoint for distinct x in F since if not we would have $y \in D_n(x, \varepsilon/2) \cap D_n(z, \varepsilon/2)$ for some $x \neq z$ in F. This would imply that $d(\alpha x, \alpha z) \leq \varepsilon$ for $\alpha \in F_n$, contradicting the existence of y.

Now $D_n(x, \varepsilon/2) = x.D_n(1, \varepsilon/2)$ since each α is a group automorphism. So we have $\mu(D_n(1, \varepsilon/2)) \leq 1/s_n(\varepsilon)$ and hence:-

$$s(\varepsilon) \leq \limsup_{n \to \infty} \{ \frac{-1}{|F_n|} \log \mu(D_n(1, \varepsilon/2))\}$$

and $h(\alpha_M) \leq \mathrm{Bow}(\alpha_M)$.

Now let $G \subset X$ be an (n, δ)-spanning set so that X is covered by the sets $D_n(x, 2\delta) = x \cdot D_n(1, 2\delta)$ with $x \in G$. So we have $D_n(1, 2\delta) \cdot r_n(\delta) \geq 1$ and:-

$$r(\delta) \geq \limsup_{n\to\infty} \{ \frac{-1}{|F_n|} \log \mu(D_n(1, 2\delta)) \}$$

and hence $h(\alpha_M) \geq Bow(\alpha_M)$ as required. □

Notice that $D_n(1, \varepsilon)$ is equal to the intersection over $\alpha \in F_n$ of the sets $\alpha^{-1}B_\varepsilon(1)$ so we have:-

$$h_{top}(\alpha) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \{ \frac{-1}{|F_n|} \log \mu(\bigcap_{T\in F_n} \alpha^{-1}B_\varepsilon(1)) \}$$

In order to analyse the global topological entropy of a $\mathbb{Z}^d$ action defined by an **R**-module **M**, we use the decomposition described in Lemma I.3.1: if **M** is a finitely generated **R**-module then there is a set of associated primes $\{\mathbf{p}_1, \ldots, \mathbf{p}_m\}$ and a chain of submodules with successive quotients isomorphic to $\mathbf{R}/\mathbf{q}_i$ with $\mathbf{q}_i$ a prime ideal containing some $\mathbf{p}_j$. We use this to express a member of *Mod(d)* as an iterated skew product of elements of *Id(d)*.

Theorem II.1.6: The system $(\mathbf{M}^*, \mathbb{Z}^d) \in$ *Mod(d)* with the chain of submodules $\mathbf{M} = \mathbf{N}_1 \supset \ldots \supset \mathbf{N}_s = \{0\}$ with quotients $\mathbf{N}_i/\mathbf{N}_{i+1} = \mathbf{R}/\mathbf{q}_i$ is metrically isomorphic to an iterated skew product of the systems $((\mathbf{R}/\mathbf{q}_i)^*, \mathbb{Z}^d) \in$ *Id(d)*.

Proof: First take the annihilators of the above chain in $X = \mathbf{M}^*$ to obtain the reverse chain:-

$$\{0\} = \mathbf{M}^{\perp} \subset X_1 \subset \ldots \subset X_s = \{0\}^{\perp} = \mathbf{M}^*, \text{ where } X_i = \mathbf{N}_i^{\perp}.$$

We will show that the first X_s can be 'factored out' in this sense and the statement will then follow by induction.

Consider the group X and the invariant subgroup $Y = X_{s-1}$. Notice that X_{s-1} is closed since it is the annihilator of a subgroup of the discrete group **M**. Let $c : X/Y \longrightarrow X$ be a Borel section of the quotient map $\pi : X \longrightarrow X/Y$. Such a map always exists, see for instance Theorem A.7 of Zimmer [1] and can in fact be chosen to be almost continuous (continuous off a null set) by Theorem 2 in Lind [3]. For the moment write X additively. We claim that each $\alpha \in \mathbb{Z}^d$ is metrically isomorphic to the skew action $\mathcal{T} = \boldsymbol{\alpha} \times_\gamma \alpha_Y$ where α_Y is the action of α on the invariant subgroup Y and $\boldsymbol{\alpha}$ is the induced action on X/Y and the skewing map $\gamma: X/Y \longrightarrow Y$ is defined by:

$$\gamma(\mathbf{x}) = \alpha(c(\mathbf{x})) - c\alpha(\mathbf{x}) \text{ where } \mathbf{x} \in X/Y.$$

Thus, $\mathcal{T}(\mathbf{x}, y) = (\boldsymbol{\alpha}(\mathbf{x}), \alpha_Y(y) + \gamma(\mathbf{x}))$. Let $\varphi : X/Y{\times}Y \longrightarrow X$ be given by $\varphi(\mathbf{x}, y) = c(\mathbf{x}) + y$. Then φ is a Borel isomorphism since c is a Borel map. Also, the following diagram commutes:-

$$\begin{tikzcd} X/Y \times Y \arrow[r, "T \times_\gamma T_Y"] \arrow[d, "\varphi"'] & X/Y \times Y \arrow[d, "\varphi"] \\ X \arrow[r, "\mathcal{T}"] & X \end{tikzcd}$$

We can continue in this way to conclude that there is a metric isomorphism $\Phi: (\mathbf{R}/\mathbf{q}_1)^* \times \ldots \times (\mathbf{R}/\mathbf{q}_s)^* \longrightarrow \mathbf{M}^*$ so the given system is metrically equivalent to an iterated skew product of elements of *Id(d)*. □

The above allows the global entropy of the system $(\mathbf{M}^*, \mathbb{Z}^d)$ to be calculated.

Corollary II.1.7: With the above notation:

$$h_M = \sum_{i=1}^{s-1} h_{R/q_i}.$$

Proof: This is an immediate consequence of the addition formula shown in Appendix B of Lind, Schmidt and Ward [1]. The case $d = 1$ was shown for the metric entropy of a skew product of endomorphisms of a Lebesgue space by Abramov and Rokhlin [1], and for a skew product of an endomorphism of a Lebesgue space and a topological group endomorphism by R. K. Thomas [1]. This gives the above topological result for $d = 1$ when combined with the fact that Haar measure is the measure of maximal entropy for group automorphisms (see Berg [1]). □

Remark II.1.8: If the system $(X_M, \mathbb{Z}^d)$ satisfies a higher dimensional analogue of Bowen's weak specification (Bowen [3], Ruelle [1] and Sigmund [1]) then there is a direct metric decomposition of an element of $\mathcal{M}od(d)$ into elements of $\mathcal{I}d(d)$. This can be demonstrated by following a method due to Doug Lind [2], [4], where it is shown for $\mathbb{Z}$ actions.

Theorem II.1.9: The topological entropy of the $\mathbb{Z}^d$ action defined by the prime ideal $\mathbf{p} \in \mathbf{R}$ is given by:-

$$h_{top}(X_{\mathbf{p}}) = \begin{cases} 0 & \text{if } \mathbf{p} \text{ is not principal} \\ \displaystyle\int_{s_1=0}^{1} \cdots \int_{s_d=0}^{1} \log |f(e^{2\pi i s_1}, \ldots, e^{2\pi i s_d})| \, ds_1 \ldots ds_d & \text{if } \mathbf{p} = \langle f \rangle, f \in \mathbf{R} \setminus \{0\} \\ \infty & \text{if } \mathbf{p} = \{0\} \end{cases}$$

Proof: This result, due to Doug Lind and Klaus Schmidt, is shown in Lind, Schmidt and Ward [1][2]. A brief description of the proof is given in Appendix B. For the case $d = 1$, this gives Yuzvinskii's formula. To see this, recall Jensen's formula: if $p(x)$ is the polynomial $a_0 + a_1 x + \ldots + a_d x^d$ and the roots of $p(z) = 0$ are $\{\lambda_i \mid i = 1, \ldots, d\}$ then:-

$$\int_0^1 \log |p(e^{2\pi i \theta})| \, d\theta = \log |a_d| + \sum_{i=1}^{d} \log \max \{|\lambda_i|, 1\}$$

[2]The formula for $h_{top}(\langle f \rangle)$ is the *logarithmic Mahler measure* of the polynomial f introduced in Mahler [1]. For certain important examples f (with zeros on the d-dimensional torus) it has been computed explicitly by C. J. Smyth and D. Boyd, see Smyth [1], [2] and Boyd [1]. The Mahler measure of a polynomial with no zeros on the d-torus is easily computed by using Jensen's formula.

which is Yuzvinskii's formula. □

Theorem II.1.9 and Corollary II.1.7 together allow the entropy of an element of *Mod(d)* to be computed. In Lind, Schmidt and Ward [1] this result is extended to compute the joint entropy of a $\mathbb{Z}^d$ action on any compact group.

§2: Periodic Points for $\mathbb{Z}^d$- actions

In order to consider periodic points for actions of $\mathbb{Z}^d$ we make the following definitions. The first, Fix(), denotes the subgroup of points of given period. The second, $\mathrm{Fix}^I()$, is a canonically chosen subgroup of the points with given period with the important property that for systems whose associated primes are non-zero there are always periods with $\mathrm{Fix}^I()$ finite. Write X_M for the compact group defined by the module **M**.

Definition II.2.1: If $\Gamma \subset \mathbb{Z}^d$ is a cocompact lattice, then $x \in X_M$ is said to be Γ-periodic if $\alpha_{\mathbf{n}}(x) = x$ for all $\mathbf{n} \in \Gamma$. Write

$$\mathrm{Fix}_M(\Gamma) = \{x \in X \mid \alpha_{\mathbf{n}}(x) = x \text{ for all } \mathbf{n} \in \mathbb{Z}^d\}.$$

If Γ is the rectangular lattice $\Gamma = n_1 e_1 \mathbb{Z} + \ldots + n_d e_d \mathbb{Z}$, then write $\mathrm{Fix}_M(\mathbf{n})$ for $\mathrm{Fix}_M(\Gamma)$; if $n_1 = \ldots = n_d$ we will call the lattice square and write $\mathrm{Fix}_M(n)$ for $\mathrm{Fix}_M(\Gamma)$.

Notice that:-

$$|\mathrm{Fix}(\mathbf{n})| = |\mathbf{R}/\langle \mathbf{q}, 1-u_1^{n_1}, \ldots, 1-u_d^{n_d} \rangle|$$

for the case $\mathbf{M} = \mathbf{R}/\mathbf{q}$ when $\mathrm{Fix}(\mathbf{n})$ is a finite group. This is because the two groups above are dual to each other and we can apply Lemma I.2.1. If either of the above quantities is infinite then so is the other - though they have different cardinalities since one of them is discrete.

Write $I(\pi) = < \varphi_{\pi_1}(u_1), \ldots, \varphi_{\pi_d}(u_d) >$, where φ_k is the k^{th} cyclotomic polynomial, and write $J(\pi) = < 1 - u_1^{\pi_1}, \ldots, 1 - u_d^{\pi_d} >$. Since $J(\pi)$ is a subgroup of $I(\pi)$, the group $\mathbf{M}/I(\pi)\mathbf{M}$ is a quotient of $\mathbf{M}/J(\pi)\mathbf{M}$. This means that the dual of $\mathbf{M}/I(\pi)\mathbf{M}$ is a closed subgroup of the dual of $\mathbf{M}/J(\pi)\mathbf{M}$. Thus all the points in $\mathrm{Fix}_{\mathbf{M}}^{I}(\pi) = (\mathbf{M}/I(\pi)\mathbf{M})^*$ have period π, and all the points with least period π lie in $\mathrm{Fix}_{\mathbf{M}}^{I}(\pi)$.

Definition II.2.2: Define the map $\Delta_X: X \longrightarrow X$ to be the group homomorphism sending the element $x \in X$ to $(\alpha_{e_1} - 1_X) \ldots (\alpha_{e_d} - 1_X) \cdot x$ where 1_X is the identity on X and $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are the usual basis elements of $\mathbb{Z}^d$. With the same notation as in Definition II.2.1 above, write $\mathrm{Fix}^{\Delta}(\mathbf{n}) = \Delta_X(\mathrm{Fix}^{I}(\mathbf{n}))$. We will write $\Delta = \Delta_{\mathbf{M}}$ when $X = X_{\mathbf{M}}$ for a finitely generated $\mathbf{R}$ - module $\mathbf{M}$.

Definition II.2.3: For the system $X_{\mathbf{M}}$ in $\mathcal{Mod}(d)$, define upper and lower growth rates of periodic points, written $H_{\mathbf{M}}$ and $H^{\mathbf{M}}$ respectively, as follows:

$$H_{\mathbf{M}} = \liminf_{\pi \longrightarrow \infty} \frac{1}{n_1 \ldots n_d} \log |\mathrm{Fix}_{\mathbf{M}}^{\Delta}(\mathbf{p})| \text{ and } H^{\mathbf{M}} = \limsup_{\pi \longrightarrow \infty} \frac{1}{n_1 \ldots n_d} \log |\mathrm{Fix}_{\mathbf{M}}^{\Delta}(\pi)|.$$

The periods π are taken to be *prime* in the sense that each π_i is an odd rational prime and, for convenience in later calculations, we also assume that all the π_i's are distinct. In the sequel, (upper or lower) growth rate of periodic points means the above limits taken along such periods, and we will *always assume that periods written* π *are of this form unless explicitly stated otherwise.* We also will always assume that a sequence of periods has the property that their fundamental regions form a Følner sequence in the acting group. Thus $\pi \longrightarrow \infty$ means that each $\pi_i \longrightarrow \infty$. Notice that for systems with finitely many fixed points (points of period $(1, 1, \ldots, 1)$) the quantities $|\mathrm{Fix}^I|$ and $|\mathrm{Fix}|$ have identical upper and lower growth rates, since Fix^I certainly contains all the points with least period π. We often assume that all our systems have finitely many points of any given period, so it makes sense to speak of the growth rate of periodic points.

Lemma II.2.4: Given any non zero ideal $\mathbf{q} \subset \mathbf{R}(d)$, there is an integer $L_{\mathbf{q}} \geq 1$ such that $V_{\mathbb{C}}(\mathbf{q}) \cap V_{\mathbb{C}}(I(\pi)) = \emptyset$ for all prime periods π with $\pi_i > L_{\mathbf{q}}$ for $i = 1, \ldots, d$.

Proof: If q contains a constant then $V_{\mathbb{C}}(q)$ is empty. So we may assume that $q \cap \mathbb{Z} = \{0\}$ and $q = \langle f_1, \ldots, f_r \rangle$. After writing all the polynomials $f_i, i = 1, \ldots, r$ in standard form, let L_q be an integer greater than the degrees of any of the f_i's. Let π have each $\pi_i > L_q$ and as usual assume that $\{\pi_1, \ldots, \pi_d\}$ is a set of distinct odd rational primes. Then if $\omega = (\omega_1, \ldots, \omega_d) \in V_{\mathbb{C}}(q) \cap V_{\mathbb{C}}(I(\pi))$, each polynomial f_j must vanish on ω, a primitive joint π^{th} root of unity[3], and so f_j must vanish on the orbit of ω under the Galois group of the $(\pi_1 \times \ldots \times \pi_d)^{th}$ cyclotomic extension of $\mathbb{Q}$. This requires in particular that $f_j(\omega_1, \ldots, \omega_{d-1}, z) = 0$ for every $\pi_d{}^{th}$ primitive unit root z. Since π_d is prime, this implies that the degree of f_j exceeds $\pi_d - 1 \geq L_q$ which contradicts the assumption that L_q exceeds the degree of f_j. □

Let M be a finitely generated R module.

Lemma II.2.5: In the group X_M, $\mathrm{Fix}_M(\pi)$ is the dual of $M/J(\pi)M$. The index of the group $\mathrm{Fix}_{R/q}{}^{\Delta}(\pi)$ in $\mathrm{Fix}_{R/q}{}^{I}(\pi)$ is bounded by $\Phi(\pi)$, a function of π with zero growth rate. This means in particular that $H_{R/q}$ is the lower growth rate of $|\mathrm{Fix}_{R/q}{}^{I}(\pi)|$ and $H^{R/q}$ is the upper growth rate of $|\mathrm{Fix}_{R/q}{}^{I}(\pi)|$.

Proof: As mentioned above, it is clear that $\mathrm{Fix}_M(\pi) \cong (M/J(\pi)M)^*$ since multiplication by u_i corresponds under duality to the action of α_{e_i}. Recall that $\mathrm{Fix}_M{}^{I}(\pi)$ is defined to be $(M/I(\pi)M)^*$.

The dual homomorphism to Δ is given by $\Delta^*f = (u_1 - 1) \ldots (u_d - 1) \cdot f$, so that $(\mathrm{Fix}_{R/q}{}^{\Delta}(\pi))^{\perp} = \{ f \in R \mid \Delta^* f \in I(\pi) + q \}$. Notice that $I(\pi) \subset (\mathrm{Fix}_{R/q}{}^{\Delta}(\pi))^{\perp}$, and so $\mathrm{Fix}_{R/q}{}^{\Delta}(\pi)$ is a subgroup of $(R/(q + I(\pi)))^*$.

In order to find the index $|\,(R/(q + I(\pi)))^*/\mathrm{Fix}_{R/q}{}^{\Delta}(\pi)\,|$ we need to find the size of the kernel K of the map $\Delta : (R/(q + I(\pi)))^* \longrightarrow (R/(q + I(\pi)))^*$. Since $\Delta = (\alpha_{e_1} - 1_X) \ldots (\alpha_{e_d} - 1_X)$ we have $|K| \leq \prod_{i=1,\ldots,d} |K_i|$ where K_i is the kernel of the map $x \longmapsto (\alpha_{e_i} - 1_X)x$ on $(R/(q + I(\pi)))^*$. Now $K_i{}^{\perp} = \langle q, u_i - 1, I(\pi) \rangle = q'$, and since we have the identity:

$$c_{\pi_i}(u) + (1 - u) + (1 - u)(1 + u) + \ldots + (1 - u)(1 + u + \ldots + u^{\pi_i - 2}) = \pi_i$$

[3] By this we mean that $\omega = (\omega_1, \ldots, \omega_d)$ has ω_i a primitive $\pi_i{}^{th}$ root of unity for $i = 1, \ldots, d$.

the constant $\pi_i \in \mathbf{q}'$. But then in $\mathbf{R}/\mathbf{q}'$ we can choose coset representatives with degrees bounded by π_j in the variables u_j for $j \neq i$, and bounded by 1 in the variable u_i. Thus:-

$$|K_i| = |K_i^{\perp\perp}| = |(\mathbf{q}')^{\perp}| = |\mathbf{R}/\mathbf{q}'| \leq (\pi_i)^{\prod_{j\neq i}\pi_j}.$$

Taking logarithms, we conclude that

$$\log|K| \leq \sum_{i=1}^{d} \log|K_i| \leq \sum_{i=1}^{d} \Big(\prod_{j\neq i} \pi_j\Big)\log(\pi_i)$$

from which we conclude that $|K|$ has zero growth rate, since

$$\frac{1}{\pi_1 \ldots \pi_d}\log|K| \leq \frac{1}{\pi_1 \ldots \pi_d}\sum_{i=1}^{d}\Big(\prod_{j\neq i}\pi_j\Big)\log(\pi_i) = \sum_{i=1}^{d}\frac{1}{\pi_i}\log(\pi_i)$$

and $(1/n)\log(n)$ converges to zero as $n \longrightarrow \infty$. This shows the lemma, where we take for $\Phi(\pi)$ the above estimate for $|K|$. □

In the above lemma it is of course the observation about growth rates that is important: for the purposes of growth rates, $\mathrm{Fix}_M{}^{\Delta}(\pi)$ and $(M/I(\pi)M)^*$ behave identically and in the sequel all computations of growth rates of $\mathrm{Fix}_M{}^{\Delta}(\pi)$ will be made by considering the more accessible group $(M/I(\pi)M)^*$. To facilitate this, write $\mathrm{Fix}_M{}^I(\pi) = (M/I(\pi)M)^* \subset \mathrm{Fix}_M(\pi)$. For clarity, we recall the definitions:

$\mathrm{Fix}_M(\pi) = (M/J(\pi)M)^*$,
$\mathrm{Fix}_M{}^I(\pi) = (M/I(\pi)M)^* \subset \mathrm{Fix}_M(\pi)$,
and $\mathrm{Fix}_M{}^{\Delta}(\pi) = \Delta(\mathrm{Fix}_M{}^I(\pi)) \subset \mathrm{Fix}_M{}^I(\pi)$.

Lemma II.2.6: If $V(\mathbf{p})$ has no unit roots (as defined in Section 1.1) then $|\mathrm{Fix}_{R/p}(\pi)| < \infty$ for all π in $\mathbb{Z}^d\backslash\{0\}$. For any non zero ideal $\mathbf{q}$, $|\mathrm{Fix}_{R/q}{}^I(\pi)| < \infty$ for all sufficiently large prime periods π.

Proof: Both of these assertions follow immediatly from Lemma 3.12 of Schmidt [1] and Lemma II.2.5. In the second, we need to choose periods π with each $\pi_i > L_q$ where L_q is chosen as in Lemma II.2.4. Lemma 3.12 of Schmidt [1] then shows that $\mathrm{Fix}_{R/q}{}^{I}(\pi)$ is finite, and so by Lemma II.2.5, $\mathrm{Fix}_{R/q}{}^{\Delta}(\pi)$ is finite as well. □

We now turn to the proof of a fundamental inequality relating periodic points and entropy. It is clear from the separated set definition of entropy that the joint topological entropy of a given action exceeds the upper growth rate of periodic points for that action if the system is expansive. What is more surprising is that the same result holds even when the action is not expansive. The method of proof is identical to the proof of Theorem 3.9 in Schmidt [1], where a criterion for expansiveness is shown.

Lemma II.2.7: Introduce the pseudo-metric[4] $d_{\mathbf{k}}$ on the group $X = X_{R/q}$ defined by:

$$d_{\mathbf{k}}(x, y) = \max_{\mathbf{n} \in \prod_{i=1}^{d} [0, k_i]} |x(\mathbf{n}) - y(\mathbf{n})| .$$

For the system $(X_{R/q}, \mathbb{Z}^d)$ there exists an $\varepsilon > 0$ (independent of $\mathbf{k}$) such that if there are only finitely many points of period $\mathbf{k}$ then for any two distinct points of period $\mathbf{k}$, x and y, $d_{\mathbf{k}}(x, y) > \varepsilon$.

Proof: Let the ideal $\mathbf{q} = \langle f_1, \ldots, f_s \rangle$ and define the constant ε as follows[5]:

[4]We are thinking of X additively now, so if $x, y \in X$ then $x(\mathbf{n})$ and $y(\mathbf{n})$ are elements of $[0, 1)$ under addition modulo 1 and $|x(\mathbf{n}) - y(\mathbf{n})|$ means the quotient metric inherited by $\mathbb{R}/\mathbb{Z}$ from the usual metric on $\mathbb{R}$. For later brevity of argument, regard the coordinate group as $[-\frac{1}{2}, \frac{1}{2})$ under addition modulo 1.

[5]For instance, if $\mathbf{q} = \langle 2+x+y, 5-yx^3 \rangle$ then the corresponding value of ε is $1/80$.

$$\varepsilon = \frac{1}{10 \times \max_{i=1,\ldots,s} \left(\sum |\text{coefficients in } f_i| + 2 \right)}.$$

Now assume that there are two points $x \neq y$ in $X_{R/q}$ which are of period $\mathbf{k}$ and which are within ε of each other under the $\mathbf{k}$ separation psuedo-metric $d_{\mathbf{k}}$. We will show that this produces a contradiction. Introduce the Banach subspace B of $l^{\infty}(\mathbb{Z}^d) = \{\text{bounded maps: } \mathbb{Z}^d \longrightarrow \mathbb{C}\}$ defined by:

$$B = \{v \in l^{\infty}(\mathbb{Z}^d) \mid f_j(\sigma_1, \ldots, \sigma_d)v = 0 \text{ for } j = 1, \ldots, s \\ \text{and } (\sigma_i^{k_i} - 1)v = 0 \text{ for } i = 1, \ldots, d\}$$

where $\sigma_1, \ldots, \sigma_d$ are the natural shift operators on $l^{\infty}(\mathbb{Z}^d)$. Now consider the element $z = x - y$ in the group X. By assumption, z has the property that each co-ordinate $z(\mathbf{n})$ in $[-\frac{1}{2}, \frac{1}{2})$ actually falls inside the interval $(-\varepsilon, \varepsilon)$. This means that the conditions[6] that z satisfies in order to lie in X and in order to have period $\mathbf{k}$ hold in $\mathbb{R}$ as well as in $\mathbb{R}/\mathbb{Z}$. That is to say, the expressions never exceed $\frac{1}{2}$ in modulus so the reduction modulo 1 never occurs. So, regarding the set $[-\frac{1}{2}, \frac{1}{2})$ as a *subset* of $\mathbb{R} \subset \mathbb{C}$ rather than a quotient *group* we can say that $z \in B$. By assumption, z is non zero so B is a non-trivial Banach space. Now define the Banach algebra $\mathcal{A}$ of (bounded linear) operators on B by setting $\mathcal{A} = \langle\{\sigma_1, \ldots, \sigma_d\}\rangle$. By general considerations (see for instance Section 7, Chapter II of Naimark [1] or Chapter 10 of Rudin [1]) there is a ring homomorphism $\eta: \mathcal{A} \longrightarrow \mathbb{C}$. Write $\omega_i = \eta(\sigma_i)$ for $i = 1, \ldots, d$. Now $f_1(\sigma_1, \ldots, \sigma_d), \ldots, f_s(\sigma_1, \ldots, \sigma_d), (\sigma_1^{k_1}-1), \ldots,$ and $(\sigma_d^{k_d}-1)$ are the zero operator on B (by definition) and so, since η is a homomorphism, $f_1(\omega_1, \ldots, \omega_d) = \ldots = f_s(\omega_1, \ldots, \omega_d) = 0$ and $(\omega_1^{k_1}-1) = \ldots = (\omega_d^{k_d}-1) = 0$. Thus $(\omega_1, \ldots, \omega_d)$ is a joint $\mathbf{k}^{\text{th}}$ unit root in $V(\mathbf{q})$, which is a contradiction of the

[6]For instance, if we consider the previous example, where $\mathbf{q} = \langle 2+x+y, 5-yx^3 \rangle$ then the conditions on z are: $2z(k, l) + z(k+1, l) + z(k, l+1) = 0$ mod. 1 and $5z(k, l) - z(k+3, l+1) = 0$ mod. 1 (in order to lie in X) and $z(k,l) = z(k+l_1,l) = z(k,l+l_2)$ mod. 1 (in order to have period l) for all $k, l \in \mathbb{Z}$. The assumption that each $z(k, l)$ lies in $(-\varepsilon, \varepsilon)$ means that the conditions are satisfied in $\mathbb{R}$: they are true modulo 1 but none of the expressions ever exceeds $\frac{1}{2}$ in modulus. In fact the constant ε is chosen to ensure that none of the expressions exceeds $1/10$ in modulus.

assumption that there were only finitely many points of this period. □

Corollary II.2.8: For any member of *Mod(d)* with non-zero associated primes, we have $\mathrm{Fix}_M{}^{\Delta}(\pi)$ finite for all large periods π, and $h_M \geq H^M$.

Proof: First we establish this for the case where $M = R/q$ for some non-zero ideal $q = \langle f_1, \ldots, f_s \rangle$. Consider two distinct points x and y in the group $\mathrm{Fix}_{R/q}{}^I(\pi)$, where π is chosen large enough to ensure that this group is finite by Lemma II.2.4. If $z = x - y$ has least period π and is within ε of the identity with respect to the π separation pseudo metric then the proof of Lemma II.2.7 shows that there must be a point in the intersection $V_{\mathbb{C}}(q) \cap V_{\mathbb{C}}(I(\pi))$. To see this, consider the point z in isolation: z has *least* period π and so it defines a non-trivial element of the Banach subspace

$$C = \{ v \in l^{\infty}(\mathbb{Z}^d) \mid f_j(\sigma_1, \ldots, \sigma_d)v = 0 \text{ for } j = 1, \ldots, s \\ \text{and } \varphi_{\pi_i}(\sigma_i)v = 0 \text{ for } i = 1, \ldots, d \},$$

and of the subspace used in Lemma II.2.7,

$$B = \{ v \in l^{\infty}(\mathbb{Z}^d) \mid f_j(\sigma_1, \ldots, \sigma_d)v = 0 \text{ for } j = 1, \ldots, s \\ \text{and } (\sigma_i^{k_i} - 1)v = 0 \text{ for } i = 1, \ldots, d \}.$$

The same argument as that used in Lemma II.2.7 now gives the point in the intersection: the fact that B is non-trivial shows there is a π^{th} unit root in the intersection; the fact that C is non-trivial shows that there must be a *primitive* π^{th} unit root in the intersection.

We deduce that x and y in $\mathrm{Fix}_{R/q}{}^I(\pi)$ are separated unless $x - y$ has a least period other than π. Since each of the π_i is prime, this can only occur if $x - y$ is actually fixed in some direction. Thus, x and y are separated unless $x - y$ lies in the kernel of the map $\Delta_{R/q}$. By choosing $S(\pi)$ to be a section of the canonical quotient map $\mathrm{Fix}_{R/q}{}^I(\pi) \longrightarrow \mathrm{Fix}_{R/q}{}^I(\pi)/(\mathrm{Fix}_{R/q}{}^I(\pi) \cap \ker\Delta)$ we therefore obtain an ε separated subset of $\mathrm{Fix}_{R/q}{}^I(\pi)$. By the definition of entropy, the growth rate of this subset is therefore dominated by the joint entropy. Now notice that the proof of

Lemma II.2.5 shows that the growth rate of the size of $S(\pi)$ is the same as the growth rate of $\mathrm{Fix}_{R/q}{}^{I}(\pi)$. This follows from the observation that the kernel of Δ on $\mathrm{Fix}_{R/q}{}^{I}(\pi)$ is bounded in size by $\Phi(\pi)$. We conclude that the growth rate of $\mathrm{Fix}_{R/q}{}^{I}(\pi)$ is dominated by the entropy, and therefore (by Lemma II.2.5 again), $H^{M} \leq h_{M}$.

We now turn to the case of a general module $\mathbf{M}$. Let the associated primes of $\mathbf{M}$ be $\{p_1, \ldots, p_m\}$ and assume that none of the ideals p_i are zero[7]. Let d be any translation invariant metric on X_M. From the definition of topological entropy in terms of separated sets, we need only check that the set $S(\pi)$ is separated for large π. We claim this: there exists an $\varepsilon > 0$ such that for any period π with $V_{\mathbb{C}}(I(\pi)) \cap \bigcup_{i=1,\ldots,m} V_{\mathbb{C}}(p_i) = \emptyset$, the set $S(\pi)$ is uniformly ε-separated. That is,

$$\sup_{n \in \mathbb{Z}^d} d(\alpha_n(x), \alpha_n(y)) > \varepsilon \text{ for any } x \neq y \text{ in } S(\pi).$$

In order to see this, assume it is not true. Then given any $\varepsilon' > 0$ we can choose a period $\pi_{\varepsilon'}$ with $V_{\mathbb{C}}(I(\pi_{\varepsilon'})) \cap \bigcup_{i=1,\ldots,m} V_{\mathbb{C}}(p_i) = \emptyset$ with the property that $S(\pi_{\varepsilon'})$ is not uniformly ε'-separated. That is, we have $d(\alpha_m x, 0) < \varepsilon'$ for all $m \in \mathbb{Z}^d$ and some $x = z - y$ for z and y distinct elements of $S(\pi_{\varepsilon'})$. From this we can find a sequence of periods $\pi^{(n)}$ such that, for every $n \geq 1$,

$V_{\mathbb{C}}(I(\pi^{(n)})) \cap \bigcup_{i=1,\ldots,m} V_{\mathbb{C}}(p_i) = \emptyset$;
$d(\alpha_m x_n, 0) < 1/n$ for all $\mathbf{m} \in \mathbb{Z}^d$ and some $x_n = z_n - y_n$ with z_n and y_n distinct points in $S(\pi^{(n)}) \backslash \{0_{X_M}\}$.

Choose a prime filtration $\{0\} = N_0 \subset N_1 \subset \ldots \subset N_r = \mathbf{M}$ with j^{th} quotient $N_j/N_{j-1} \cong R/q_j$, where q_j is a prime ideal in $\mathbf{R}$ containing $p_{k(j)}$, one of the associated primes of $\mathbf{M}$. Since $V_{\mathbb{C}}(q_j) \subset V_{\mathbb{C}}(p_{k(j)})$, we certainly have:

$$V_{\mathbb{C}}(I(\pi^{(n)})) \cap \bigcup_{i=1,\ldots,m} V_{\mathbb{C}}(q_j) = \emptyset \text{ for all } n \geq 1 \text{ and } j = 1, \ldots, r.$$

[7]We keep assuming this. The reason is that if $\mathbf{M}$ has $\{0\}$ as an associated prime, then (cf. Section I.3) there is a filtration of $\mathbf{M}$ with one quotient isomorphic to $R/\{0\} \cong R$. Corresponding to this, there is an invariant subgroup of X_M isomorphic to the dual of R - a full d-dimensional shift with alphabet $\mathbb{T}$. This guarantees that the system has infinite entropy and infinitely many points of any given period. The groups Fix^{Δ} are also infinite in this case.

In the filtration, N_j is a subgroup of M so $N_j^{\perp}$ is a subgroup of $M^* = X_M$. Furthermore, $X_M/N_j^{\perp} \cong M^*/N_j^{\perp} \cong N_j^* \cong X_{N_j}$. Taking the annihilator in X_M of the terms in the filtration produces a chain of invariant subgroups of X_M: $X_M = N_0^{\perp} \supset N_1^{\perp} \supset \ldots \supset N_r^{\perp} = \{0_{X_M}\}$. Since $(N_j/N_{j-1})^* \cong N_{j-1}^{\perp} \subset N_j^* \cong M^*/N_j^{\perp}$, $(N_j/N_{j-1})^* \cong N_{j-1}^{\perp}/N_j^{\perp} \cong (R/q_j)^* \cong X_{R/q_j}$.

Write $\eta_j: X_M \longrightarrow X_M/N_j^{\perp} \cong X_{N_j}$ for the quotient map. Since η_j^*, the dual of η_j, is a homomorphism of commutative $\mathbf{R}$-modules , we have $\eta_j(x_n) \in \mathrm{Fix}_{N_j}{}^{I}(\pi^{(n)})$ for all integers $n \geq 1$. We conclude that $\eta_1(x_n) \in \mathrm{Fix}_{N_1}{}^{I}(\pi^{(n)})$, and since $N_1/N_0 \cong N_1 \cong R/q_1$, Lemmas II.2.7 and II.2.6 together imply that there is an N_1 such that x_n is in the kernel of the map Δ_{N_1/N_0} for $n \geq N_1$, since it will be too close to the identity. So, $x_n \in \ker(\Delta_{N_1/N_0})$ for every $n \geq N_1$. This means that z_n and y_n lie in the same coset of $\ker\Delta$ on N_1/N_0.

Consider $N_1^{\perp}/N_2^{\perp}$. As mentioned above, $N_1^{\perp}/N_2^{\perp} \cong X_{R/q_2}$, so we can apply Lemmas II.2.6 and II.2.7 to the sequence $(\eta_2(x_n))_{n \geq N_1}$ to conclude that there is an N_2 with $x_n \in \ker(\Delta_{N_2/N_1})$ for every $n \geq N_2$. By repeating this argument a total of r times we obtain an integer N_r such that x_n lies in the kernel of Δ on $N_r^{\perp} = \{0_{X_M}\}$ for every $n \geq N_r$ and so z_n and y_n lie in the same coset of $\ker\Delta$ on $\mathbf{M}$. Since z_n and y_n are in the section $S(\pi)$, they must therefore be equal, contradicting the assumption that $x_n \in \mathrm{Fix}_M{}^{I}(\pi^{(n)})\backslash\{0_{X_M}\}$ for all $n \geq 1$, so we conclude the existence of a separation constant ε with the stated properties. By compactness, this also shows that for large periods π, $\mathrm{Fix}_M{}^{I}(\pi)$ is finite.

The proof is completed by showing that the growth rate of the set $S(\pi)$ is correct. Each $S(\pi)$ is the image of a section of the quotient map $\mathrm{Fix}_M{}^{I}(\pi)$ onto $\mathrm{Fix}_M{}^{I}(\pi)/(\mathrm{Fix}_M{}^{I}(\pi) \cap \ker\Delta)$. So the size is given by the image of Δ, to which we may apply the same argument as was used in the proof of Lemma II.2.5 as follows. First notice that $|\ker(\Delta \text{ on } \mathbf{M}/I(\pi)\mathbf{M})| \leq \prod_{i=1,\ldots,d} |\ker(u_i - 1) \text{ on } \mathbf{M}/I(\pi)\mathbf{M})|$. Now apply the filtration to conclude that $|\ker(u_i - 1) \text{ on } \mathbf{M}/I(\pi)\mathbf{M})| = \prod_{i=1,\ldots,s} |\ker(u_i - 1)$ on $\mathbf{R}/I(\pi)\mathbf{R}|$. This finally gives the estimate $|\ker(\Delta \text{ on } \mathbf{M}/I(\pi)\mathbf{M})| \leq [\Phi(\pi)]^s$ which has zero growth rate since Φ itself has zero growth rate. □

§3: Growth Rate of Periodic Points and Topological Entropy

In this section we wish to show that the growth rate of periodic points coincides with the joint topological entropy for $\mathcal{M}od(d)$. We start with an example due to Doug Lind showing that such a statement cannot be possible in the category of all actions on compact abelian groups. The regularity implied by the Descending Chain Condition is essential - without it we may have no non-trivial periodic points despite the presence of positive entropy.

Example II.3.1: (See example after Proposition 3.1 in Lind and Ward [1][8]) Consider initially the (discrete) group $D = \mathbb{Z}[1/6]$ and the automorphism $A = [3/2]$ acting on $X = D^*$. The entropy is given by Abramov's formula, $h(A) = \log 3$. We will consider the same system below in Lemma II.3.2 with $f(u) = 3 - 2u$, and so, anticipating the result there, we expect many periodic points. In fact, the closure of the subgroup of A-periodic points has annihilator:-

$$\bigcap_{n=1}^{\infty}[(3/2)^n-1]D = \{0\}$$

That is, the periodic points of A are dense in X. Now introduce isometric components as follows. Let $D_{\mathbb{Q}} = D\otimes_{\mathbb{Z}}\mathbb{Q} = \mathbb{Q}$, and let $X_{\mathbb{Q}} = D_{\mathbb{Q}}{}^*$. The map A extends to an automorphism $A_{\mathbb{Q}}$ of $X_{\mathbb{Q}}$. Notice that the resulting $\mathbb{Z}$-action does not satisfy the descending chain condition. We still have $h(A_{\mathbb{Q}}) = \log 3$ but the closure of the subgroup of $A_{\mathbb{Q}}$-periodic points has annihilator:-

$$\bigcap_{n=1}^{\infty}[(3/2)^n-1]D_{\mathbb{Q}} = D_{\mathbb{Q}}$$

Thus $A_{\mathbb{Q}}$ has only the identity as a periodic point. To see that $(X_{\mathbb{Q}}, A_{\mathbb{Q}})$ does not have the descending chain condition, define the sets X_i as follows. Let X_i be the annihilator of $\mathbb{Z}[1/P_i]$ where P_i is the set of the first $(i+2)$ rational primes. So, for

[8]This example is given in Lind and Ward [1] to show the effect on a solenoidal automorphism of introducing components that do not have hyperbolicity.

instance, $P_3 = \{2, 3, 5, 7, 11\}$ and $X_3 = \mathbb{Z}[1/2310]$. We then have $X_{\mathbb{Q}} \supset X_1 \supset X_2 \supset \ldots$ a chain of $A_{\mathbb{Q}}$-invariant closed subgroups that does not terminate. Notice that the X_i have the finite intersection property but the intersection of all the X_i is $\{1_X\}$.

We now turn to the problem of computing the growth rate of periodic points for elements of $\mathcal{Mod}(d)$. The method involves induction on the dimension of the acting group and a reduction to *Id(d)*. There is an inherent difficulty in that the $\mathbb{Z}^{d-1}$ action induced on the subgroup of points of period n in the e_d direction of a system in *Id(d)* is not an element of *Id(d-1)*: it is in general an element of $\mathcal{Mod}(d\text{-}1)$.

To start the induction, we begin by considering *Id(1)*. Here the growth rate of periodic points for an ergodic $\mathbb{Z}$-action is equal to the entropy. This is well known, since the system has Markov partitions by Fried [1]. For a general discussion of this case see Kitchens and Schmidt [1], Example 12.1(3) and Section 6, or Lawton [1]. Notice that if $|f_0| = |f_n| = 1$ in the discussion below then (X_f, α_f) is an ergodic automorphism of a torus. For the case of an expansive automorphism of a torus, Lemma II.3.2 is shown in Theorem 8.18 of Walters [1]. This case also follows in part from topological dynamics: an expansive automorphism of the torus is an axiom A map of a compact manifold, so Bowen [1] shows that the topological entropy is the upper (limsup) growth rate of periodic points. The case of an ergodic automorphism of the torus is dealt with in Lind [4]. In general, the group X_f is a direct product of a restricted solenoid[9] and a full shift whose alphabet is a finite cyclic group.

Lemma II.3.2: Let $(X_\mathbf{p}, \mathbb{Z})$ be an ergodic member of *Id(1)*. Then the growth rate of periodic points is equal to the topological entropy of α_f on X_f.

Proof: We may assume that the ideal $\mathbf{p}$ is principal, say $\mathbf{p} = \langle f \rangle$. Notice that the corresponding group $X_f = X$ is a solenoid of the type found in Lawton [1]. By multiplying by a unit in $\mathbf{R}$, that is to say a monomial, we can assume that:–

[9] An automorphism of a full solenoid i.e. the action of an element of $GL(n,\mathbb{Q})$ on $(\mathbb{Q}^*)^n$ is never expansive (see Lind and Ward [1]) because there are always infinitely many isometric directions corresponding to all the rational primes that do not appear in the numerator or denominator of any element of the rational matrix or in the determinant of the rational matrix. For instance, the solenoidal automorphism determined by the matrix $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 1 \end{bmatrix}$ has an isometric direction corresponding to each rational prime $\neq 2, 3$ or 5.

$$f(u) = f_0 + f_1u + \ldots + f_nu^n \quad (f_0f_n \neq 0).$$

Let s be the highest common factor of $f_0, \ldots, f_n$. Then there is an isomorphism $\psi: X_f \longrightarrow (\mathbb{Z}/s\mathbb{Z}) \times X_{(f/s)}$ which carries α_1 to $\sigma \times \alpha_1(f/s)$ where σ is the shift on s symbols. Let A^t be the companion matrix to the polynomial $g = f/f_n$. Then (see 12.1(3) of Kitchens and Schmidt [1]) the system $(X_{(f/s)}, \alpha_1(f/s))$ is algebraically conjugate to the system $(A, Y(A, f_n/s))$ where $Y(A, f_n/s)$ is the solenoid defined by the integer f_n/s and the rational matrix A:-

$$Y(A,f_n/s) = \{x \in (\mathbb{T}^n)^{\mathbb{Z}} \mid (f_n/s)x(k) = (f_n/s)Ax(k+1)\}$$

The number of points of period v is therefore given by the expression $|\mathrm{Fix}_v(\sigma)| \times |\mathrm{Fix}_v(\alpha_1(f/s)|$ which is:-

$$|\mathrm{Fix}(T_1(f))| = s^v \times (f_n/s)^v \times \det(A^v - I) = |f_n^v| \times \prod_{i=1}^{n} |\lambda_i^v - 1|$$

where the λ_i's are the eigenvalues of A, none of which are unit roots by ergodicity. It can then be shown[10] that the growth rate of this number is:-

$$\lim_{n\to\infty} \frac{1}{v} \log |\mathrm{Fix}(\alpha_1^v(f))| = \log |f_n| + \sum_{|\lambda_i| \geq 1} \log |\lambda_i|$$

This is Yuzvinskii's formula so we have shown that in this case the growth rate of periodic points is equal to the entropy. This formula is shown in Yuzvinskii [1] and in Lind and Ward [1][11]. □

[10]As pointed out by Doug Lind in the toral case Lind [4] the possibility of eigenvalues λ with $|\lambda|=1$ makes this convergence non-trivial. In this case the automorphism is said to be *quasihyperbolic*. However, such a λ is not a unit root by ergodicity and Gelfond's theorem Gelfond [1] can be applied: $|\lambda^k - 1| > e^{-\varepsilon k}$ for only finitely many k for each $\varepsilon > 0$. This is equivalent to the required convergence.

[11]This formula was found by Arov [1] for the case where the characteristic polynomial is a power of an irreduciblepolynomial, and by Abramov [1] when the matrix is 1×1 (see Examples II.5.1 *(v)* for a description of an analogous system in higher dimensions). Yuzvinskii

In order to carry out the reduction in dimension, we will need to compute the growth rate of periodic points for the $\mathbb{Z}^{d-1}$ action induced on an element of *Id(d)* when we restrict to points of given period n in the last direction. It turns out that the resulting systems are particularly nice elements of $\mathcal{M}od(d\text{-}1)$: they are determined by modules of the form $\mathbf{M}_A = \mathbf{R}(d-1)^n/A\mathbf{R}(d-1)^n$ where A is an $n \times n$ matrix with entries in $\mathbf{R}(d-1)$. These systems are of independent interest, initially because the entropy of the actions associated to them can be computed without finding all their associated primes. To start with, we find the growth rate of periodic points for such a system.

Lemma II.3.3: Let A be an element of $M_n(\mathbf{R}(d))$ with non-zero determinant. Then, if $\mathbf{M}_A$ is the module $\mathbf{R}(d)^n/A\mathbf{R}(d)^n$, the growth rate of periodic points in the system $(X_{\mathbf{M}_A}, \mathbb{Z}^d)$ is given by $H^{\mathbf{M}_A} = \log M(\det(A))$, where $M(f)$ is as usual the Mahler measure of the polynomial $f \in \mathbf{R}(d)$.

Proof: For brevity write $\mathbf{M} = \mathbf{M}_A$. By Lemma II.2.4 there is an integer $N \geq 1$ such that $V_{\mathbb{C}}(\langle\det(A)\rangle) \cap V_{\mathbb{C}}(I(\pi)) = \emptyset$ for all periods π with each $\pi_i \geq N$. Fix such a period π and write $D = D(\pi) = (\pi_1 - 1) \times \ldots \times (\pi_d - 1)$. Under the standing assumption that the components of the period π are distinct rational primes, D is the degree of the extension $\mathbb{Q}(\omega):\mathbb{Q}$ where $\omega = e^{2\pi i/\pi_1 \ldots \pi_d}$. Write $k = \mathbb{Q}(\omega)$.

Write adj(A) for the classical adjoint of A. Then there is the usual identity $A\cdot\operatorname{adj}(A) = \operatorname{adj}(A)\cdot A = \det(A)\cdot I_n$, where I_n is the identity matrix in $M_n(\mathbf{R})$. In particular, this shows that $\det(A)\cdot\mathbf{R}^n \subset A\cdot\mathbf{R}^n$, so $\det(A)\cdot\mathbf{M} = \{0_{\mathbf{M}}\}$. This means that if $\mathbf{p}$ is a prime ideal associated with $\mathbf{M}$ then $\mathbf{p} \supset \langle\det(A)\rangle$, and if $\mathbf{p} = \langle g\rangle$ is a principal prime ideal associated with $\mathbf{M}$ then g divides det(A). Notice that for every prime $\mathbf{p}$ associated with $\mathbf{M}$ we have $V_{\mathbb{C}}(\mathbf{p}) \subset V_{\mathbb{C}}(\langle\det(A)\rangle)$ so $V_{\mathbb{C}}(\mathbf{p}) \cap V_{\mathbb{C}}(I(\pi)) = \emptyset$ and this means that $\mathrm{Fix}_{\mathbf{M}}{}^{\Delta}(\pi)$ is finite by Corollary II.2.8.

Write $\Omega(\pi) = \{ \boldsymbol{\omega} = (\omega_1, \ldots, \omega_d) \mid \omega_i$ is a primitive $\pi_i{}^{\text{th}}$ unit root for $i = 1, \ldots, d. \}$. For each $\boldsymbol{\omega} \in \Omega(\pi)$, let $\Pi(\boldsymbol{\omega}) = \omega_1 \times \ldots \times \omega_d$. Notice that $\Pi(\boldsymbol{\omega})$ is a

[1] derives the entropy directly in this form using some complicated linear algebra. In Lind and Ward [1] a covering-space method is used and it is shown that the entropy is given by $\sum \log|\lambda_{i,p}|_v$ where the sum is taken over all inequivalent places (finite and infinite) of $\mathbb{Q}$, and the $\{\lambda_{i,p}\}$ are the roots of f viewed as a polynomial over $\mathbb{Q}_p$ where p is the prime lying below the place v.

primitive $(\pi_1 \times \ldots \times \pi_d)^{th}$ unit root. Define the evaluation homomorphism at ω, $\eta_\omega: R \longrightarrow \mathbb{Z}[\Pi(\omega)] \cong \mathbb{Z}^D$ by $\eta_\omega(f) = f(\omega)$. The surjective map $\eta_\omega \times \ldots \times \eta_\omega$ from R^n to $\mathbb{Z}[\Pi(\omega)]^n$ induces a surjective homomorphism η_ω from $M = R^n/AR^n$ to $\mathbb{Z}[\Pi(\omega)]^n/A(\omega)\mathbb{Z}[\Pi(\omega)]^n$ where $A(\omega)$ is the matrix in $M_n(\mathbb{Z}[\Pi(\omega)])$ whose $(i, j)^{th}$ entry is $\eta_\omega(a_{i,j})$. Since the kernel of η_ω is the ideal $I(\pi)$, the kernel of η_ω is $I(\pi)M$ and so

$$\eta_\omega(M) = \mathbb{Z}[\Pi(\omega)]^n/A(\omega)\mathbb{Z}[\Pi(\omega)]^n \cong M/I(\pi)M = (\mathrm{Fix}_M{}^I(\pi))^*.$$

Thus, in order to compute $|\mathrm{Fix}_M{}^I(\pi)|$ we need to find $|\mathbb{Z}[\Pi(\omega)]^n/A(\omega)\mathbb{Z}[\Pi(\omega)]^n|$. We already know that $|\mathrm{Fix}_M{}^I(\pi)| < \infty$ so $|\mathrm{Fix}_M{}^I(\pi)|$ is the absolute value of the determinant of $A(\omega)$ acting as a linear map on $\mathbb{Z}[\Pi(\omega)]^n$.

We claim that $|\det(A(\omega))| = |\prod_{\Omega(\pi)} \det(A)(\sigma)|$ where the produce is taken over all $\sigma \in \Omega(\pi)$. Notice that the absolute value of the determinant of $z \in \mathbb{Z}[\Pi(\omega)]$ acting on $\mathbb{Z}[\Pi(\omega)]^n$ is $N_{k:\mathbb{Q}}(z) = \prod_\gamma \gamma(z)$ where the product is taken over all $\gamma \in \mathrm{Gal}(k:\mathbb{Q})$ (see for instance Weil [1]), and $\mathrm{Gal}(k:\mathbb{Q})$ is transitive on $\Omega(\pi)$. For any matrix $X \in M_n(R)$ write $d(X) = |\det(X(\omega))|$ and $\delta(X) = |\prod_{\Omega(\pi)} \det(X)(\sigma)|$. For two matrices B and C in $M_n(R)$ it is clear that $d(BC) = d(B) \times d(C)$ since $(BC)(\omega) = B(\omega)C(\omega)$. Similarly, we have the identity $\delta(BC) = \delta(B) \times \delta(C)$ since $\det(BC) = \det(B) \times \det(C)$. This means we need only check that $d = \delta$ for elementary matrices since any element of $M_n(k)$ can be written as a product of elementary matrices[12].

(i) Let $B = \begin{bmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{bmatrix}$ be a diagonal matrix. Then $B(\omega)$ is the diagonal matrix with entries $(b_1(\omega), \ldots, b_n(\omega))$. So

$$d(B) = \prod_{i=1,\ldots,n} N_{k:\mathbb{Q}}(b_i(\omega)) = \prod_{i=1,\ldots,n}(\prod_\sigma(b_i(\sigma))) = \prod_\sigma(\det(B))(\sigma) = \delta(B).$$

[12]Notice we do the reduction to a product of elementary matrices in the field k after evaluating on the joint unit root. It is not possible to do this reduction over the ring R: the ring R is not a GE-ring (generalized euclidean). This is shown in §5 of Cohn [1].

(ii) Let $B = \begin{bmatrix} 1 & & & & \\ & \cdot & & & \\ & & 1 & 1 & \\ & & & \cdot & \\ & & & 1 & \\ & & & & \cdot \\ & & & & & 1 \end{bmatrix}$ be a row addition matrix. Then it is clear that $d(B) = \delta(B) = 1$.

(iii) Let $B = \begin{bmatrix} 1 & & & & \\ & \cdot & & & \\ & & 0 & 1 & \\ & & & \cdot & \\ & & 1 & 0 & \\ & & & & \cdot \\ & & & & & 1 \end{bmatrix}$ be a permutation matrix. Then it is again clear that $d(B) = \delta(B) = 1$.

So d and δ are multiplicative functions coinciding on the elementary matrices. We conclude in particular that $d(A) = \delta(A)$.

The growth rates H_M and H^M are then given by the limit as π goes to infinity of

$$\frac{1}{\pi_1 \ldots \pi_d} \log |\mathrm{Fix}_M^I(\pi)| = \frac{1}{\pi_1 \ldots \pi_d} \sum_{\sigma \in \Omega(\pi)} \log(|\det(A)(\sigma)|).$$

the last term is a Riemann approximation to the logarithmic Mahler measure $\log M(\det(A))$. The result follows since H^M is the growth rate of $|\mathrm{Fix}_M{}^I(\pi)|$ by Lemma II.2.5.

If there are no zeros of $\det(A)$ on the unit torus[13] then the sequence converges and we may conclude that $H_M = H^M$. □

Call the above result $S(d)$. Call the statement $h = H$ for systems determined by prime ideals in $Id(d)$, $P_{Id(d)}$, and $h = H$ for $\mathcal{M}od(d)$, $P_{\mathcal{M}od(d)}$. Notice that $P_{Id(1)}$ is

[13]This corresponds to assuming that the action on X_A is expansive by Theorem 3.8 of Schmidt [1]. In this case, the sequences converge without any restriction on the periods other than requiring that they become large in each direction, which is the Følner condition. The expansive case is discussed in Lind, Schmidt and Ward [1].

shown in Lemma II.3.2. We can then represent the induction argument schematically as follows:

$$P_{\mathit{Id}(1)};\ P_{\mathit{Id}(d)} \Longrightarrow P_{\mathcal{M}od(d)}\ ;\ P_{\mathcal{M}od(d)} \text{ and } S\,(d) \Longrightarrow P_{\mathit{Id}(d+1)}.$$

Theorem II.3.4: If $h_{R/q} = H_{R/q} = H^{R/q}$ for all non zero ideals $q \subset R(d)$, then $h_M = H_M = H^M$ for any finitely generated $R(d)$-modules M none of whose associated primes are zero.

Proof: This is the step $(P_{\mathit{Id}(d)} \Longrightarrow P_{\mathcal{M}od(d)})$ in the scheme above. The proof is long and involves reduction to primary modules; we break it into three parts.

Step One. Let $q \subset R$ be a non-zero ideal which does not contain any constants ($q \cap \mathbb{Z} = \{0\}$), and let $g \in R \setminus q$. Then if $K \subset L$ are finitely generated q-primary modules with $gL \subset K$ we have $h_K = h_L$, $H_K = H_L$, and $H^K = H^L$.

Proof. Fix a period π with each $\pi_i \geq L_q$, where L_q is chosen as in Lemma II.2.4. Let $M \subset N$ be finitely generated q-primary modules with $gN \subset M$. Then

$$|N/I(\pi)\cdot N| \leq |N/(M + I(\pi)\cdot N)| \times |(M + I(\pi)\cdot N)/I(\pi)\cdot N|.$$

Consider the term $|N/(M + I(\pi)\cdot N)|$. For each $x \in N/M$, $(q + \langle g\rangle)x \in M$ since $gN \subset M$ so none of the associated primes of N/M are principal (recall that we are assuming q is prime so $g \in R \setminus q$ implies that $\langle g\rangle + q$ is not principal). By Theorem II.1.9 and Corollary II.2.8, this means that

$$\begin{aligned}
\limsup_{\pi\to\infty} \frac{1}{\pi_1 \cdots \pi_d} \log |N/(M+I(\pi)N)| &= \limsup_{\pi\to\infty} \frac{1}{\pi_1 \cdots \pi_d} \log |\mathrm{Fix}^{\Delta}_{N/M}(\pi)| \\
&= \limsup_{\pi\to\infty} \frac{1}{\pi_1 \cdots \pi_d} \log |\mathrm{Fix}^{I}_{N/M}(\pi)| \\
&= H^{N/M} \leq h_{N/M} = 0.
\end{aligned}$$

Now consider the second term, $(\mathbf{M} + I(\pi)\cdot\mathbf{N})/I(\pi)\cdot\mathbf{N}$. Since $I(\pi)\cdot\mathbf{M} \subset I(\pi)\cdot\mathbf{N}$, $|(\mathbf{M}+I(\pi)\cdot\mathbf{N})/I(\pi)\cdot\mathbf{N}| \le |\mathbf{M}/I(\pi)\cdot\mathbf{M}|$ and so we may conclude that

$$H_N \le H_M \text{ and } H^N \le H^M. \qquad (*)$$

By the Addition Formula from Appendix B of Lind, Schmidt and Ward [1], we also have $h_N = h_M + h_{N/M} = h_M$.

Now assume that $\mathbf{M} = g\mathbf{N}$. The homomorphism $\mathbf{N} \longrightarrow \mathbf{M}/I(\pi)\cdot\mathbf{M}$, given by sending $a \longmapsto ga + I(\pi)\cdot\mathbf{M}$ is surjective, and the kernel contains $I(\pi)\cdot\mathbf{N}$ by commutativity. This means in particular that $|\mathbf{M}/I(\pi)\cdot\mathbf{M}| \le |\mathbf{N}/I(\pi)\cdot\mathbf{N}|$, and so $H_M \le H_N$. By the previous paragraph, we then have

$$H_N = H_{gN} \text{ and } H^N = H^{gN}. \qquad (\dagger)$$

By applying $(*)$ with $\mathbf{M} = \mathbf{K}$ and $\mathbf{N} = \mathbf{L}$ and then with $\mathbf{M} = g\cdot\mathbf{L}$ and $\mathbf{N} = \mathbf{K}$, we get that $h_L = h_K$, $H_L \le H_K \le H_{gL}$, and $H^L \le H^K \le H^{gL}$. But $(\dagger)$ implies that $H_{gL} = H_L$ and $H^{gL} = H^L$, and so Step One is completed. □

Step Two. Let $\mathbf{q} \subset \mathbf{R}$ be a non-zero ideal which does not contain any constants $(\mathbf{q} \cap \mathbb{Z} = \{0\})$, and let $\mathbf{N}$ be a finitely generated $\mathbf{q}$-primary module. If $h_{R/q} = H_{R/q}$ then $h_N = H_N = H^N$.

Proof. If $\mathbf{q}$ is not principal then by Lind, Schmidt and Ward [1] $h_N = 0$ and so $H_N = H^N = 0$ by Corollary II.2.8 and we are done.

Assume that $\mathbf{q} = \langle f \rangle$ is a principal ideal. We can choose[14] an adapted filtration $\{0\} = \mathbf{N}_0 \subset \mathbf{N}_1 \subset \ldots \subset \mathbf{N}_r = \mathbf{N}$ with the property that $\mathbf{N}_j/\mathbf{N}_{j-1} \cong \mathbf{R}/\mathbf{q}_j = \mathbf{R}/\mathbf{q}$ for $i = 1, \ldots, s \le r$ and $\mathbf{N}_j/\mathbf{N}_{j-1} \cong \mathbf{R}/\mathbf{q}_j$ for some prime ideal $\mathbf{q}_j$ with $\mathbf{q} \subsetneq \mathbf{q}_j$ for $i = s+1, \ldots, r$. For every $i = 1, \ldots, s$ choose an irreducible polynomial $g_i \in \mathbf{q}_i \setminus \mathbf{q}$ and let $g = g_1 \cdot \ldots \cdot g_s$. Then $g \notin \mathbf{q}$ since $\mathbf{q}$ is prime, and $g\cdot\mathbf{N} \subset \mathbf{N}_s \subset \mathbf{N}$ and so by Step One we have $h_{N_s} = h_N$, $H_{N_s} = H_N$, and

[14] This may be seen as follows. The prime ideal $\mathbf{q}$ is an associated prime of $\mathbf{N}$ so there is an element $a \in \mathbf{N}$ with $\mathrm{Ann}(a) = \mathbf{q}$. Put $\mathbf{N}_1 = a\cdot\mathbf{R}$. Then the map $\mathbf{R} \longrightarrow \mathbf{N}_1$ is surjective and has kernel $\mathbf{q}$ so $\mathbf{N}_1/\{0\} \cong \mathbf{R}/\mathbf{q}$. Now consider the quotient $\mathbf{N}/\mathbf{N}_1$. This either has $\mathbf{q}$ as an associated prime, in which case the next element of the filtration may be produced in the same way, or has not - in which case we are done, since all the quotients above the point reached are of the form $\mathbf{R}/\mathbf{q}'$ for some $\mathbf{q}'$ with $\mathbf{q} \subsetneq \mathbf{q}'$.

$H^{N_s} = H^N$. This means that we can assume without loss in generality that $s = r$ and so $q_i = q = \langle f \rangle$ for all $i = 1, \ldots, r$.

Choose elements $a_i \in N$ with $N_i = R \cdot a_i + N_{i-1}$ and $q = \{h \in R \mid f \cdot a_i \in N_{i-1}\}$ for each i. Denote by $\kappa_i : R/q \longrightarrow N_i/N_{i-1}$ the usual isomorphism given by $\kappa_i(h + q) = h \cdot a_i$.

Let π be a period with[15] $I(\pi) \cap q \neq I(\pi) \cdot q$. Since $I(\pi) \cap q \supset I(\pi) \cdot q$ this can only occur if there is a polynomial $h \in (I(\pi) \cap q) \setminus I(\pi) \cdot q$, and this implies that one of the irreducible factors of h lies in $I(\pi) \cap q$. But $q = \langle f \rangle$ for some irreducible polynomial f so we must have $f \in I(\pi)$. This requires $V_{\mathbb{C}}(q) \supset V_{\mathbb{C}}(I(\pi))$, and hence $V_{\mathbb{C}}(q) \cap V_{\mathbb{C}}(I(\pi)) \neq \emptyset$. By applying Lemma II.2.4 we conclude that there is a constant L_q such that $I(\pi) \cap q = I(\pi) \cdot q$ for all periods π with every $\pi_i \geq L_q$.

Fix a period π with each $\pi_i \geq L_q$. We claim that $N_j \cap I(\pi) \cdot N = I(\pi) \cdot N_j$ for every $i = 1, \ldots, r$. In order to see this, assume that $I(\pi) \cdot N_j \subsetneq N_j \cap I(\pi) \cdot N$ for some $j \in \{1, \ldots, r\}$. Then $(N_j \cap I(\pi) \cdot N)/(N_{j-1} + I(\pi) \cdot N_j) \neq \{0\}$ which means that we may choose an $x \in (N_j \cap I(\pi) \cdot N) \setminus (N_{j-1} + I(\pi) \cdot N_j)$. Denote by $t > j$ the smallest integer with the property that $x \in I(\pi) \cdot N_t$. Then, by assumption, we can find $x_1, \ldots, x_s \in N_t$ and $f_1, \ldots, f_s \in I(\pi)$ with $x = f_1 \cdot x_1 + \ldots + f_s \cdot x_s$ and $\{x_1, \ldots, x_s\} \not\subset N_{t-1}$. The isomorphism $\kappa_t : R/q \longrightarrow N_t/N_{t-1}$ ($h + q \longmapsto h \cdot a_t + N_{t-1}$) then lets us choose $g_1, \ldots, g_s \in R$ with $y_i = x_i - g_i a_t \in N_{t-1}$ for $i = 1, \ldots, s$. Since we are assuming $\{x_1, \ldots, x_s\} \not\subset N_{t-1}$ we must have $\{g_1, \ldots, g_s\} \not\subset q$, but

$$f_1 g_1 \cdot a_t + \ldots + f_s g_s \cdot a_t = f_1(x_1 - y_1) + \ldots + f_s(x_s - y_s) \in N_{t-1}.$$

This means that $f_1 g_1 + \ldots + f_s g_s \in q \cap I(\pi)$ (since $\sum f_i g_i$ is in the kernel of κ_t). Now if $f_1 g_1 + \ldots + f_s g_s \in q . I(\pi)$ then there exist polynomials $f'_1, \ldots, f'_{s'} \in I(\pi)$ and $g'_1, \ldots, g'_{s'} \in q$ with

$$f_1 g_1 + \ldots + f_s g_s = f'_1 g'_1 + \ldots + f'_{s'} g'_{s'}$$

and (writing $z_i = g'_i \cdot a_t \in N_{t-1}$)

[15] This is then a bad period in terms of lifting periodic points (see Section II.6). The situation is this: we have a filtration of the form $M \supset N \supset \{0\}$, and we would like to have $H^M = H^N + H^{M/N}$. It is clear that $|M/IM| = |N/(IM \cap N)| \times |(M/N)/(IM/N)|$ where we have written I for $I(\pi)$. Thus, if $IM \cap N = IN$ for all large periods, we get $H^M = H^N + H^{M/N}$ immediatly.

$$x = (f'_1 z_1 + \ldots + f'_{s'} z_{s'}) + (f_1 y_1 + \ldots + f_s y_s) \in I(\pi)\cdot N_{t-1}.$$

This contradicts the assumption that t is the least integer with $x \in I(\pi)\cdot N_t$, and it therefore follows that $f_1 g_1 + \ldots + f_s g_s \in \mathbf{q} \cap I(\pi) \setminus \mathbf{q}.I(\pi)$ which contradicts the assumption that π is chosen to have $\mathbf{q} \cap I(\pi) = \mathbf{q}.I(\pi)$.

Having verified that $N_i \cap I(\pi) = I(\pi)\cdot N_i$ for all $i = 1, \ldots, r$ and sufficiently large periods π, we conclude that

$$|\mathrm{Fix}_{N_i/N_{i-1}} I(\pi)| = |N_i/(N_{i-1} + I(\pi)\cdot N_i)| = |N_i/(N_{i-1} + I(\pi)\cdot N)|$$

for $i = 1, \ldots, r$. Then

$$|\mathrm{Fix}_N^{\Delta}(\pi)| = |N/I(\pi)N| = \prod_{i=1}^{r} |N_i/(N_{i-1} + I(\pi)N| = \prod_{i=1}^{r} |N_i/(N_{i-1} + I(\pi)N_i|$$
$$= \prod_{i=1}^{r} |\mathrm{Fix}^{\Delta}_{N_i/N_{i-1}}(\pi)| = |\mathrm{Fix}^{\Delta}_{R/q}(\pi)|^r.$$

By Lind, Schmidt and Ward [1] we know that $h_N = r\cdot h_{R/q}$, from the above we have $H_N = r\cdot H_{R/q}$ and $H^N = r\cdot H^{R/q}$ by Lemma II.2.5 and our hypothesis $h_{R/q} = H_{R/q}$ then gives

$$h_N = r\cdot h_{R/q} = r\cdot H_{R/q} = H_N \le H^N \le h_N$$

by Corollary II.2.8. □

<u>Step Three</u>. Proof of Theorem II.3.4.

Let $\mathbf{M}$ be a finitely generated $\mathbf{R}$-module with $h_M < \infty$ and associated primes $\{\mathbf{p}_1, \ldots, \mathbf{p}_s\}$. None of the associated primes are zero by the assumption on the entropy. Choose a reduced primary decomposition (see Section 1.3) of 0_M. That is a collection of submodules $\mathbf{M}_1, \ldots, \mathbf{M}_s$ with $\mathbf{M}/\mathbf{M}_i$ $\mathbf{p}_i$-primary for $i = 1, \ldots, s$ and $\mathbf{M}_1 \cap \ldots \cap \mathbf{M}_s = \{0\}$. The map $\theta : \mathbf{M} \longrightarrow (\mathbf{M}/\mathbf{M}_1) \oplus \ldots \oplus (\mathbf{M}/\mathbf{M}_s)$ defined by $\theta(a) = (a + \mathbf{M}_1, \ldots, a + \mathbf{M}_s)$ is an injective homomorphism since the kernel is

$M_1 \cap \ldots \cap M_s = \{0\}$. Write $X = X_M$ and $X_i = X_{M/M_i}$ for $i = 1, \ldots, s$. Since θ is injective, the dual map $\eta = \theta^* : X_1 \times \ldots \times X_s \longrightarrow X$ is surjective. Set $Y = \ker(\eta)$, and notice that Y^* is isomorphic to the cokernel of θ. That is, if we write $\mathbf{N} = \theta(\mathbf{M}) \subset (\mathbf{M}/\mathbf{M}_1) \oplus \ldots \oplus (\mathbf{M}/\mathbf{M}_s)$, then the duality results of Section I.2 show that $Y^* = \mathbf{K} = ((\mathbf{M}/\mathbf{M}_1) \oplus \ldots \oplus (\mathbf{M}/\mathbf{M}_s))/\mathbf{N}$. By the Addition Formula of Lind, Schmidt and Ward [1] we have:

$$h_{(M/M_1) \oplus \ldots \oplus (M/M_s)} = h_{M/M_1} + \ldots + h_{M/M_s} = h_M + h_K$$

and we claim that $h_K = 0$. To see this, renumber the associated primes of $\mathbf{M}$ so that $\mathbf{p}_1, \ldots, \mathbf{p}_r$ are principal and $\mathbf{p}_{r+1}, \ldots, \mathbf{p}_s$ are non-principal. Let $a \in \mathbf{M}$ and put $\mathbf{a}_i = (0, \ldots, 0, a+\mathbf{M}_i, 0, \ldots, 0) \in (\mathbf{M}/\mathbf{M}_1) \oplus \ldots \oplus (\mathbf{M}/\mathbf{M}_s)$. If $i > r$ then the ideal $\{f \in \mathbf{R} \mid f^k \cdot \mathbf{a}_i \in \mathbf{N} \text{ for some } k \geq 1\}$ contains $\mathbf{p}_i$ and is therefore non-principal. If $i \leq r$ then $\{ f \in \mathbf{R} \mid f^k \cdot \mathbf{a}_i \in \mathbf{N} \text{ for some } k \geq 1 \}$ contains $\mathbf{p}_i + \prod_{j \neq i} \mathbf{p}_j$ and is again non-principal since the associated primes are distinct. From this it follows at once that all the associated primes of $\mathbf{K}$ are non-principal, and so we have $h_K = 0$, and thus

$$h_M = h_{M/M_1} + \ldots + h_{M/M_s}.$$

Now by Step Two and the assumption we are making in Theorem II.3.4, $h_{M/M_i} = H_{M/M_i}$ for $i = 1, \ldots, s$. Also, since $h_K = 0$, $H_K = 0$ by Corollary II.2.8. Then

$$h_M = h_{M/M_1} + \ldots + h_{M/M_s} = H_{M/M_1} + \ldots + H_{M/M_s} \leq H_M + H_K = H_M \leq H^M \leq h_M$$

which proves the theorem. □

Theorem II.3.5: Let $\mathbf{M}$ be any finitely generated $\mathbf{R}(d)$ - module with all of its associated primes non-zero. Then $h_M = H_M = H^M$.

Proof: We need to show the step ($P_{\mathcal{M}od(d)}$ and $S(d) \Longrightarrow P_{Id(d+1)}$) in the scheme above and then use induction.

The theorem holds for dimension one by Lemma II.3.2. Assume $P_{\mathcal{M}od(d)}$: the theorem holds for dimension d. We need to show $P_{Id(d+1)}$. So, let $\mathbf{p}$ be a prime ideal in $\mathbf{R}(d+1)$, and consider the system $(X = X_{\mathbf{R}/\mathbf{p}}, \alpha_{\mathbf{R}/\mathbf{p}})$. If $\mathbf{p}$ is not principal then $h_{\mathbf{R}/\mathbf{p}} = H_{\mathbf{R}/\mathbf{p}} = H^{\mathbf{R}/\mathbf{p}}$ by Theorem II.1.9 and Corollary II.2.8. So assume that $\mathbf{p} = \langle f \rangle$ for some irreducible polynomial $f \in \mathbf{R}(d+1)\setminus\{0\}$. Fix some rational prime $\pi = \pi_{d+1}$ exceeding the degree of f and let $X^{\pi} = \{x \in X \mid \alpha_{(0, 0, \ldots, 0, \pi)}(x) = x\}$ denote the subgroup of points with period π in the last direction, and let $Y^{\pi} = \Delta_X(X^{\pi})$. By defining $\alpha'_{\mathbf{n}}(x) = \alpha_{(n_1,\ldots,n_d,0)}(x)$ for every $\mathbf{n} \in \mathbb{Z}^d$ we induce an action α' of $\mathbb{Z}^d$ on X^{π} and by defining $\alpha^{\Delta}_{\mathbf{n}}(x) = \alpha_{(n_1,\ldots,n_d,0)}(x)$ we induce an action α^{Δ} on Y^{π}.

Define the $\mathbf{R}(d)$ module $\mathbf{M}(\pi) = \mathbf{S}/\mathbf{B}\mathbf{S}$ where $\mathbf{S}$ and $\mathbf{B}$ are defined as follows. Write f as $f_0 + u_{d+1}f_1 + \ldots + u_{d+1}{}^n f_n$ with $f_i \in \mathbf{R}(d)$ and $f_0 f_n \neq 0$ - multiply by some power of u_{d+1} if needed. Let $A \in M_{\pi}(\mathbf{R}(d))$ be the π^{th} circulant matrix of f with respect to the variable u_{d+1}, defined as follows:

$$A = \begin{bmatrix} f_0 & f_1 & & & f_n & & \\ & f_0 & f_1 & & & f_n & \\ & & \ddots & & & & \ddots \\ & & & f_0 & f_1 & & f_n \\ f_n & & & & f_0 & f_1 & f_{n-1} \\ \vdots & & & & \vdots & \ddots & \vdots \\ & \ddots & & & & & \\ & & & f_n & & & f_0 \end{bmatrix}.$$

Set $\mathbf{S} = \{(h_0, \ldots, h_{\pi-1}) \in \mathbf{R}(d)^{\pi} \mid h_0 + \ldots + h_{\pi-1} = 0\} \cong \mathbf{R}(d)^{\pi-1}$. This is an A-invariant submodule of $\mathbf{R}(d)^{\pi}$ and we denote by B the restriction of A to $\mathbf{S}$. We then have:

$(X^{\pi}, \alpha') \cong (X_{\mathbf{R}(d)^{\pi}/A\mathbf{R}(d)^{\pi}}, \alpha_{\mathbf{R}(d)^{\pi}/A\mathbf{R}(d)^{\pi}})$

and $(Y^{\pi}, \alpha^{\Delta}) \cong (X_{\mathbf{M}(\pi)}, \alpha_{\mathbf{M}(\pi)})$.

By the usual theory of circulants, we have:

$$\det(A)(u_1, \ldots, u_d) = \prod_{k=0,\ldots,\pi-1} f(u_1, \ldots, u_d, \omega^k)$$

and

$$\det(B)(u_1, \ldots, u_d) = \prod_{k=1,\ldots,\pi-1} f(u_1, \ldots, u_d, \omega^k)$$

where ω is a primitive π^{th} unit root. Notice that $\det(B) \neq 0$. We include $\det(A)$

for interest only; the point of using Δ, $\mathbf{B}$ and $\mathbf{S}$ is to avoid the need to use $\det(A)$ which in many interesting examples is zero for all periods.

By assumption $P_{\mathcal{M}od(d)}$, we have $h_{M(\pi)} = H_{M(\pi)} = H^{M(\pi)}$. This, together with Lemma II.3.3, implies that:

$$\liminf_{\pi\to\infty} \frac{1}{\pi} H_{M(\pi)}$$

$$= \liminf_{\pi\to\infty} \frac{1}{\pi} \int_0^1 \dots \int_0^1 \log\Big(\prod_{k=1}^{\pi-1} f(e^{2\pi i s_1}, \dots, e^{2\pi i s_d}, \omega^k) \Big)\, ds_1 \dots ds_d$$

$$= \liminf_{\pi\to\infty} \frac{1}{\pi} \sum_{k=1}^{\pi-1} \int_0^1 \dots \int_0^1 \log| f(e^{2\pi i s_1}, \dots, e^{2\pi i s_d}, \omega^k)|\, ds_1 \dots ds_d$$

$$= \int_0^1 \dots \int_0^1 \log| f(e^{2\pi i s_1}, \dots, e^{2\pi i s_{d+1}})|\, ds_1 \dots ds_{d+1} = h_M$$

by Lind, Schmidt and Ward [1]. Now Lemma II.2.5 and the induction hypothesis show that:

$$\liminf_{\pi\to\infty} \frac{1}{\pi} H_{M(\pi)} = \liminf_{\pi\to\infty} \frac{1}{\pi} H^{M(\pi)} = \liminf_{\pi\to\infty} \frac{1}{\pi_1 \dots \pi_{d+1}} \log|\mathrm{Fix}^{\Delta}_M(\pi)| = H_M .$$

The last two equalities show $P_{Id(d+1)} : h_M = H_M = H^M$. By Theorem II.3.4 we conclude $P_{\mathcal{M}od(d+1)}$ as required. □

The above proof can be slightly shortened by applying Lemma II.3.3 to the 1×1 matrix [f]. We have chosen not to do this in order to illuminate the relationship between the properties of circulants and the Riemann approximations.

For expansive actions the results of this section will appear in Lind, Schmidt and Ward [1]. Expansiveness considerably simplifies the situation. In particular, the growth rate of periodic points sequence converges along any sequence of periods that has fundamental regions forming a Følner sequence.

§4: Examples and Remarks

In this section we give some explicit values for the topological entropy of some simple low-dimensional systems, and compute directly the growth rate of periodic points for a few of the examples. Many more examples are described in Lind, Schmidt and Ward [1].

Examples II.4.1: Using the formula for the topological entropy of the system $(X_{<f>}, \mathbb{Z}^d)$ given in Lind, Schmidt and Ward [1] we have the following.

(i) Let $f(x, y) = 1 + x + y$. Then (quoted in Boyd [1], due to Smyth [2]):

$$h_{top}(X_{<f>}) = \frac{3\sqrt{3}}{4\pi} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^2} \quad \text{where } \chi(n)=0, 1, -1 \text{ as } n=0, 1, 2 \bmod 3.$$

(ii) Let $f(x, y, z) = 1 + x + y + z$. Then (quoted in Boyd [1], due to Smyth [2]):

$$h_{top}(X_{<f>}) = \frac{7}{2\pi^2}\zeta(3) = \frac{7}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^3}$$

(iii) Let[16] $f(x, y) = x^2 - y^2 + xy + 3x - y + 1$. Then (Smyth [2]):

$$h_{top}(X_{<f>}) = 2\cdot\log(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}).$$

(iv) Let $f(u_1, \ldots, u_d)$ be a finite product of monomials $\mathbf{u}^{\mathbf{n}}$ (units in R) and polynomials of the form $\varphi(\mathbf{u}^{\mathbf{n}})$ where φ is cyclotomic. Then (and only then) the topological entropy is zero[17]. (see Smyth [1], Boyd [2]).

[16] This example is one of a certain class of polynomials in several variables considered in Smyth [2] whose Mahler measure is an algebraic number (recall the entropy is the logarithm of the Mahler measure). For polynomials in one variable over the integers this is always the case by Jensen's formula. For general polynomials over the integers it is not known when the logarithmic Mahler measure is of this form.

(v) (Two dimensional Abramov[18] system) Let $f(x, y) = a + bx + cy$. Then if $V(f)$ does not contain any joint unit roots:

$$h_{top}(X_{\langle f\rangle}) = \log \max \{|a|, |b|, |c|\}$$

(vi) Let $f(x, y) = y - g(x)$ and assume that $|g(z)| = 1$ has no roots of unit modulus (this means the corresponding $\mathbb{Z}^2$ action is expansive and by Kronecker's theorem in Kronecker [1] we must have $|g(z)| > 1$ for all z with $|z| = 1$). Then:

$$h_{top}(X_{\langle y-g(x)\rangle}, \mathbb{Z}^2) = h_{top}(X_{\langle g\rangle}, \mathbb{Z}) = \log |d| + \sum_{|\lambda_i| \geq 1} \log |\lambda_i|$$

where d is the leading coefficient of g and $\{\lambda_i\}$ are the roots of $g(z) = 0$. The expansiveness condition is essential here. By Jensen's formula:

$$h_{top}(X_{\langle y-g(x)\rangle}, \mathbb{Z}^2) = \int_0^1\int_0^1 \log |e^{2\pi is} - g(e^{2\pi it})| \, ds dt = \int_0^1 \max \log\{1, g(e^{2\pi it})\} \, dt$$

Now by the expansiveness condition and Kronecker, this is just $\int_0^1 \log |g(e^{2\pi it})| \, dt$ which is Yuzvinskii's formula by Jensen's formula again.

For the last two examples above we directly compute the growth rate of periodic points.

Examples II.4.2: Using the elimination method we calculate the periodic points in $X_{\langle f\rangle}$.

[17]This is an extension due to Boyd and Smyth of Kronecker's classical theorem Kronecker [1]: if $Q(z)$ is a monic polynomial with integer coefficients, $Q(0) \neq 0$ and all the roots of $Q(z)=0$ lie in $\{z \mid |z| \leq 1\}$ then they are all roots of unity.

[18]For the one-dimensional case $f(x)=a+bx$, with a, b coprime Abramov [1] computed the topological entropy $h_{top}(X_{\langle a+bx\rangle}) = \log \max \{|a|, |b|\}$.

(i) Let $f(x, y) = a + bx + cy$. Then the subgroup of points of period m in the y direction is the solenoid defined by the polynomial $f_m(x) = (cx+a)^m - (-b)^m$. The number of points of period n in this one-dimensional system is given by:

$$|\mathrm{Fix}(n,m)| = (c^m)^n \times \prod_{j=1}^{m} \left| \frac{(-1)^m}{(c^n)} (a+be^{2\pi ij/m})^n - 1 \right| = \prod_{j=1}^{m} \left|(a+be^{2\pi ij/m})^n - (-c)^n\right|$$

The growth rate of this quantity is $\log \max \{|a|, |b|, |c|\}$ so long as there are no joint unit roots.

(ii) Let $f(x, y) = y - g(x)$. Let the leading coefficient of g be d and let the roots of $g(z) = 1$ be $\{\lambda_i\}_{i=1,\ldots,v}$. To find the points of period (n,m), eliminate y from the pair of polynomials $\{1 - y^m, f(x, y)\}$. This gives the polynomial $[g(x)]^m - 1$. The zeros of this polynomial are $\{e^{2\pi ij/m}\lambda_i \mid j=1, \ldots, m;\ i=1, \ldots, v\}$ (since $|g(z)| - 1$ has no zeros of modulus one) and so the points of period n in the solenoid defined by $g^m - 1$ number:

$$|\mathrm{Fix}(n,m)| = |d|^{nm} \times \prod_{i=1}^{v} \prod_{j=1}^{m} \left|e^{2\pi ijn/m} \lambda_i^n - 1\right| = |d|^{nm} \times \prod_{i=1}^{v} \left|\lambda_i^{nm} - 1\right|$$

The growth rate of this is then the Yuzvinskii formula:

$$\log d + \sum_{|\lambda_i| \geq 1} \log |\lambda_i| = h_{top}(X_{<g>}, \mathbb{Z}) = h_{top}(X_{<y-g(x)>}, \mathbb{Z}^2)$$

Remark II.4.3: From Example II.4.1 *(i)* above, we have the limit formula along some subsequence:

$$\lim_{n\to\infty} \frac{1}{(2n+1)^2} \log \prod_{j=0}^{2n} \left| (1+e^{2\pi i j/(2n+1)})^{(2n+1)} - 1 \right| = \frac{3\sqrt{3}}{4\pi} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^2}$$

which this chapter interprets as a Riemann sum.

Remark II.4.4: Theorem II.3.5 holds for actions of $\mathbb{Z}^d$ by homeomorphisms of a compact metric space X when the action is expansive and has strong specification in the sense of Ruelle [1]. The one-dimensional case of this statement can be found in Denker, Grillenberger and Sigmund [1], Chapter 22 and in Bowen [4]. The problem with this approach is twofold: firstly, the problem of deciding whether a given $\mathbb{Z}^d$ action has strong specification is open even for simple examples, and, secondly, many systems cannot have strong specification because strong specification implies mixing of all orders.

As an application of Theorem II.3.5 we isolate the following result. This result is included because it is an entropy computation that is easily done using periodic points and is more difficult to do directly.

Theorem II.4.5: For any matrix $A \in M_n(\mathbf{R}(d))$, the dynamical system (X_M, α_M) where $\mathbf{M} = \mathbf{R}(d)^n / A{\cdot}\mathbf{R}(d)^n$ has entropy given by:

$$h(\alpha_M) = \int_0^1 \cdots \int_0^1 \log|\det(A)(e^{2\pi i s_1}, \ldots, e^{2\pi i s_d})| \, ds_1 \ldots ds_d$$

if $\det(A) \neq 0$ and $h(\alpha_M) = \infty$ if $\det(A) = 0$.

Proof: The first formula is an immediate corollary of Theorem II.3.5 and Lemma II.3.3. If $\det(A) = 0$ then there is a non zero vector $\mathbf{v} \in \mathbf{R}(d)^n$ with $f{\cdot}\mathbf{v} \notin A{\cdot}\mathbf{R}(d)^n$ for every $f \in \mathbf{R}(d) \setminus \{0\}$. But this means that $\mathbf{v} + A{\cdot}\mathbf{R}(d)^n \in \mathbf{M}$ has annihilator $\{0\}$ so $\mathbf{M}$ has $\{0\}$ as an associated prime, and $h_{\mathbf{M}} = \infty$ by Lind, Schmidt and Ward [1]. □

§5: Cohomology and Periodic Points

In this section we explain some of the difficulties encountered with periodic points (see Step Two of Theorem II.3.4) in terms of a cohomology group associated to the period, and show that the error involved in assuming that all periodic points can be lifted arises from a certain subgroup of the first cohomology group of the action on the subsystem determined by the quotient module. This identifies elements of an abelian group that are *obstructions* to the lifting of periodic points. This does not help to get round the difficulties but provides a convenient framework for describing them. Recall the problem of lifting periodic points: we start with a short exact sequence of **R**-modules $0 \longrightarrow \mathbf{N} \longrightarrow \mathbf{M} \longrightarrow \mathbf{L} \longrightarrow 0$ (arising from the filtration $\mathbf{M} \supset \mathbf{N} \supset \{0\}$ of **M**, and writing **L** for the quotient module **M/N**). Dual to this[19] there is the short exact sequence $0 \longrightarrow X_L \longrightarrow X_M \longrightarrow X_N \longrightarrow 0$ of $\mathbb{Z}^d$ actions. We wish to factor out by the submodules giving the points with rectangular period **l**, namely $J(\mathbf{l})\mathbf{P} = \langle 1-u_1^{l_1}, \ldots, 1-u_d^{l_d}\rangle \mathbf{P}$, where **P** is any of the modules **L**, **M**, and **N**. This produces a sequence that is no longer exact, and so we cannot conclude that $|\mathbf{M}/J(\mathbf{l})\mathbf{M}| = |\mathbf{N}/J(\mathbf{l})\mathbf{N}| \times |\mathbf{L}/J(\mathbf{l})\mathbf{L}|$, or, dually, that $|\mathrm{Fix}_M(\mathbf{l})| = |\mathrm{Fix}_N(\mathbf{l})| \times |\mathrm{Fix}_L(\mathbf{l})|$. In this section we will put this problem in a slightly different framework.

Fix the rectangular period **l** throughout this section, and consider the exact sequence arising from the filtration $\mathbf{M} \supset \mathbf{N} \supset \{0\}$ of **M**:

$$0 \longrightarrow X_L \longrightarrow X_M \longrightarrow X_N \longrightarrow 0.$$

Define the subgroup $\Gamma = \Gamma(\mathbf{l})$ of $\mathbb{Z}^d$ defined by this period: $\Gamma(\mathbf{l}) = \mathbb{Z}.l_1 e_1 + \ldots + \mathbb{Z}.l_d e_d$.

Definition II.5.1: Let **A** be a finitely generated **R** module. A 1-cocycle (or *crossed homomorphism*) for X_A with respect to $\Gamma = \Gamma(\mathbf{l})$ is a (Borel) measurable map $\sigma: \Gamma \longrightarrow X_A$ with the property that $\sigma(\mathbf{n}+\mathbf{m}) = \sigma(\mathbf{n}) + T_{\mathbf{n}}\sigma(\mathbf{m})$ for all **n** and **m** in Γ. The collection of all the 1-cocycles forms an abelian group, called $Z^1(\Gamma, X_A)$.

[19] Recall that the dual of a homomorphism $\theta: G \longrightarrow H$ is a homomorphism $\theta^*: H^* \longrightarrow G^*$. The map θ is injective (surjective) if and only if the dual θ^* is surjective (injective). Furthermore, if there is a short exact sequence of the form $0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0$ then the dual is $0 \longleftarrow H^* \longleftarrow G^* \longleftarrow (G/H)^* \longleftarrow 0$ which is also short exact since $(G/H)^* \cong H^{\perp} \subset G^*$ and $G^*/H^{\perp} \cong H^*$.

For each point $x \in X_A$, we have a 1-cocycle σ_x defined by $\sigma_x(n) = T_n x - x$. It is clear that this is a 1-cocycle since

$$\sigma_x(n+m) = T_{(n+m)}x - x = T_n(T_m x - x) + T_n x - x = \sigma_x(n) + T_n\sigma_x(m).$$

The set $\{\sigma_x \mid x \in X_A\}$ forms a subgroup of $Z^1(\Gamma, X_A)$, denoted by $B^1(\Gamma, X_A)$. These are the 1-coboundaries, and the equivalence relation on $Z^1(\Gamma, X_A)$ defined by this subgroup is that of being (1)-*cohomologous.* The first cohomology group of X_A with respect to Γ is defined to be the group of cohomology classes of cocycles: $H^1(\Gamma, X_A) = Z^1(\Gamma, X_A) / B^1(\Gamma, X_A)$.

Proposition II.5.2: If $0 \longrightarrow X_L \longrightarrow X_M \longrightarrow X_N \longrightarrow 0$ is an exact sequence of R-modules, then $0 \longrightarrow \mathrm{Fix}_L(1) \longrightarrow \mathrm{Fix}_M(1) \longrightarrow \mathrm{Fix}_N(1)$ is also exact but the last map is not in general surjective. However, we have an exact sequence:

$$0 \longrightarrow \mathrm{Fix}_L(1) \longrightarrow \mathrm{Fix}_M(1) \longrightarrow \mathrm{Fix}_N(1) \longrightarrow H^1(\Gamma(1), X_L).$$

Proof: The first observation is clear: simply restrict the maps to the points of period 1. Denote by i and π the given monomorphism $X_L \longrightarrow X_M$ and epimorphism $X_M \longrightarrow X_N$ (and their restrictions to points of period 1) respectively. Identify X_L with its image $i(X_L)$ so that i becomes an inclusion. We construct a homomorphism $\delta\colon \mathrm{Fix}_N(1) \longrightarrow H^1(\Gamma(1), X_L)$ as follows. Given $x \in \mathrm{Fix}_N(1)$, choose $y \in X_M$ with $\pi(y) = x$, and define the map $\sigma_y\colon \Gamma \longrightarrow X_L$ by $\sigma_y(n) = T_n y - y$. Notice that this does map into X_L since $\pi(T_n y - y) = T_n x - x = 0$ for all $n \in \Gamma$, and so $T_n y - y \in \mathrm{kernel}(\pi) = X_L$. Associate to $x \in \mathrm{Fix}_N(1)$ the cohomology class $\delta(x) = \sigma_y + B^1(\Gamma, X_L)$. The map δ is well defined because if z is another element of X_M with $\pi(z) = \pi(y) = x$ then the cocycles σ_z and σ_y differ only by the cocycle $\sigma_{(z-y)}$ which is a coboundary because $z - y \in L$. To see the exactness at $\mathrm{Fix}_N(1)$, let $\delta(x) = 0$. This means that for any $y \in X_M$ with $\pi(y) = x$, the cocycle σ_y is a coboundary, so there is an element $z \in X_L$ with $\sigma_y(n) = \sigma_z(n)$ for all $n \in \Gamma$, which means that $T_n y - y = T_n z - z$ for all $n \in \Gamma$, so the point $(y - z)$ has period 1. Thus $(y - z) \in \mathrm{Fix}_M(1)$ has $\pi(y - z) = x$ and $\mathrm{kernel}(\delta) \subset \mathrm{image}(\pi)$. The reverse inclusion is clear: if $x \in \mathrm{Fix}_M(1)$ then $\sigma_x = \delta\pi(x)$ is the zero element because

$\sigma_x(n) = T_n x - x = 0$ for all $n \in \Gamma$. □

Corollary II.5.3: Assuming that the system X_M has only finitely many points of period 1, we have:

$$|Fix_M(1)| = |Fix_L(1)| \times |Fix_N(1)| \,/\, |\delta(Fix_N(1))|$$

and therefore:

$$|Fix_N(1)| \times |Fix_L(1)| \,/\, |H^1(\Gamma(1), X_L)| \leq |Fix_M(1)| \leq |Fix_N(1)| \times |Fix_L(1)|.$$

Proof: Since we are assuming that everything is finite, Lemma I.2.1 may be applied. In order to see the relationship, notice that (with the notation of Proposition II.5.2)

$$\begin{aligned} |Fix_M(1)| &= |Fix_L(1)| \times |\pi(Fix_M(1))| = |Fix_L(1)| \times |\mathrm{kernel}(\delta)| \\ &= |Fix_L(1)| \times |Fix_N(1)| \,/\, |\mathrm{image}(\delta)| = |Fix_L(1)| \times |Fix_N(1)| \,/\, |\delta(Fix_N(1))| \end{aligned}$$

by exactness. The result follows since $1 \leq |\, \delta(Fix_N(1))| \leq |H^1(\Gamma, X_L)|$. □

Many of the problems encountered in Section II.3 can now be summed up in a single question: how big is $\delta(Fix_N(1)$? The results of Section II.3 show that this group is small in that the growth rate of $|\delta(Fix_N(1))|$ is zero.

III: Algebraic Entropy

§1: Topological Entropy and Algebraic Entropy

In this section we introduce the notion of algebraic entropy for $\mathbb{Z}^d$ actions on discrete abelian groups. This was introduced for $\mathbb{Z}$ actions on torsion groups by Weiss [1], who showed that the algebraic entropy of an automorphism of an abelian torsion group is equal to the topological entropy of the dual automorphism of the (zero-dimensional) dual group. Justin Peters [1] showed that the algebraic entropy of an automorphism α of any discrete abelian group Γ is equal to the topological entropy of the dual automorphism α^* acting on the compact abelian dual group Γ^*. We show that the same result holds for $\mathbb{Z}^d$ actions. The method is identical to that of Peters. This means that the topological entropy of a system $(X_M, \mathbb{Z}^d)$ can be expressed in four distinct ways: directly as in Definition II.1.3, locally as in Definition II.1.4, via periodic points as in Theorem II.4.5 and algebraically as below. The description using periodic points is also purely algebraic: the growth rate of periodic points is equal to the growth rate of the cardinality of certain factor groups of the module **M**. It would therefore be nice to have a direct algebraic demonstration that the algebraic entropy coincides with the growth rate of periodic points. Algebraic entropy will be defined with respect to a given Følner sequence in the acting group.

Definition III.1.1: Let Γ be a discrete abelian group carrying a $\mathbb{Z}^d$ action defined by the map $S: \mathbb{Z}^d \longrightarrow \mathrm{Aut}(\Gamma)$. Given a Følner sequence F_n in $\mathbb{Z}^d$ and a finite set $E \subset \Gamma$, let $F_n(E)$ denote the set $+_{s\in F_n} S^{-1}(E)$. Define, analogously to equation (4) in Peters [1], the algebraic entropy of the action S to be:

$$h_{alg}(\Gamma) = \sup_E \limsup_{n\to\infty} \frac{1}{|F_n|} \log |F_n(E)|$$

The supremum is taken over all finite sets. We write $h_{alg}(\Gamma)$ because the actions we are interested in are determined by the groups on which they act.

As pointed out by Peters, this expression measures the amount of expansiveness at infinity in $\mathbb{Z}^d$, while the local expression $\mathrm{Bow}(X = \Gamma^*)$ with $T = S^*$ measures the degree of expansiveness of the dual action at the identity. It is therefore reasonable to regard these two expressions as dual and this is the content of Theorem III.1.2 below.

Theorem III.1.2: $h_{alg}(\Gamma) = \mathrm{Bow}(X)$, where we have taken $X = \Gamma^*$ and the action T on X to be the dual of the action S on Γ.

The proof of this result depends on rewriting $h_{alg}(\Gamma)$ and $\mathrm{Bow}(X)$ in a form that makes the duality clear.

A function φ on a locally compact group G is said to be *positive definite* if for any collections $(x_n)_{1,\ldots,N}$ in G and $(c_n)_{1,\ldots,N}$ in $\mathbb{C}$, $\sum_m \sum_n \varphi(x_n - x_m) c_m \overline{c_n} \geq 0$. If φ is a positive definite function then (see Reiter [1], IV.4.6) for any $x \in G$, $|\varphi(x)| \leq \varphi(1_G)$.

If G is a locally compact group, let $P(G)$ denote the space of continuous positive definite functions on G. Let $L^1(G)^+$ denote the set of non-negative integrable functions on G. We recall the Fourier Inversion Theorem (see Hewitt and Ross [1]):-

$$\{\varphi^* \mid \varphi \in P(G) \cap L^1(G)^+\} = P(G^*) \cap L^1(G^*)^+$$

where G^* is the Pontryagin dual of G. Recall the notation in Definition II.1.4.

Define the quantity $h^1(X)$ as follows:-

$$h^1(X) = \sup_{\varphi} \limsup_{n\to\infty} \frac{-1}{|F_n|} \log \left[(\varphi(1))^{-|F_n|} \int_X \prod_{F_n} (\varphi T)(y) \, d\mu(y) \right]$$

where the supremum is taken over all functions φ in $P(X) \cap L^1(X)^+$.

Lemma III.1.3: $\mathrm{Bow}(X) = h^1(X)$

Proof: Given any compact neighbourhood U of $1 \in X$, there is a function φ in $P(X) \cap L^1(X)^+$ with $\mathrm{Support}(\varphi) \subset U$. Then, since positive definite functions attain their maxima at the identity, $\varphi \le \varphi(1) \times \chi_U$. Notice that $\chi_{TU} = \chi_U T^{-1}$ so this means that $\varphi T \le \varphi(1) \times \chi_U T$. The product over F_n will satisfy the same inequality so:-

$$[\varphi(1)]^{-|F_n|} \int_X \prod_{T \in F_n} (\varphi T)(y) d\mu(y) \le \int_X \prod_{T \in F_n} (\chi_U T)(y) d\mu(y)$$

and hence $h^1(X) \ge \mathrm{Bow}(X)$, since $\int \prod (\chi_U T)(y) d\mu(y) = \mu(\bigcap T^{-1}U)$.

For the reverse inequality, given any $\varphi \in P(X) \cap L^1(X)^+$ and $\varepsilon > 0$, let V be the set $\{x \in X \mid (1+\varepsilon)\varphi(x) \ge \varphi(1)\}$. Then V is a compact neighbourhood of 1 by continuity and:-

$$\frac{(1+\varepsilon)^{|F_n|}}{\varphi(1)^{|F_n|}} \int_X \prod_{T \in F_n} (\varphi T)(y) d\mu(y) \ge \mu\left(\bigcap_{T \in F_n} T^{-1} V\right)$$

which implies that $h^1(X) \le \mathrm{Bow}(X) + \log(1+\varepsilon)$ and so $h^1(X) = \mathrm{Bow}(X)$. □

Lemma III.1.4: For any family (A_n) of symmetric subsets of Γ, indexed by $n \in F$, some finite subset of $\mathbb{Z}^d$, and any integer $r > 0$:-

$$\frac{\prod_{n \in F} |A_n|^2}{\underset{n \in F}{*} (\chi_{A_n} * \chi_{A_n})(0)} \le \left|\sum_{n \in F} (2A_n)\right| \le \frac{\prod_{n \in F} |(r+1)A_n|^2}{\underset{n \in F}{*} (\chi_{rA_n} * \chi_{rA_n})(0)}$$

Proof of first inequality. Write $*$, $\sum$ and $\prod$ to mean $*_{n \in F}$, $\sum_{n \in F}$ and $\prod_{n \in F}$.

Claim that $(\ast\chi_{A_n})(x)$ is the number of $|F|$-tuples $(y_n)_{n\in F}$ with $x = \sum_F y_n$. This can be seen by an induction argument on $|F|$. If $|F| = 1$ then it is clear, so we assume that it is true for any F with $|F| < N$. Let $F = F'\cup\{m\}$ say. Consider the following expression:-

$$\ast\chi_{A_n}(x) = \int_\Gamma \underset{n\in F'}{\ast}\chi_{A_n}(y) \times \chi_{A_m}(x-y)d|y| \qquad (*)$$

Notice that for a given y, $\chi_{A_m}(x-y)$ is 1 if and only if there is a y_m in A_m with $x = y + y_m$. Also, $\ast_{n\in F'}\chi_{A_n}(y)$ is 0 if $y \notin \sum_{F'}A_n$ and is the number of $|F'|$-tuples summing to y if $y \in \sum_{F'}A_n$ by the induction assumption. So $(*)$ is the number of $|F|$-tuples summing to x as required.

Now $|\sum(2A_n)|$ is the number of distinct points of the form $x = \sum(y_n+y'_n)$; $\prod|A_n|^2$ is the total number of $2|F|$-tuples $[(x_n, y_n)]_{n\in F}$ with $x_n, y_n \in A_n$. Thus:-

$$\frac{\prod_{n\in F}|A_n|^2}{\sup_{x\in R/I} \ast(\chi_{A_n} * \chi_{A_n})(x)} \leq |\sum(2A_n)|$$

Now notice that each A_n is symmetric so by a standard argument, $\ast(\chi_{A_n}*\chi_{A_n})$ is a positive definite function and hence achieves its maximum at $x = 0$. This therefore gives the first inequality.

Proof of second inequality. Set $E_F = \{ (y_n, y'_n) \mid n \in F,\ y_n, y'_n \in rA_n$ and $\sum(y_n+y'_n) = 0 \}$. For each $x \in \sum(2A_n)$ there are at least $|E_F|$ $2|F|$-tuples in $\prod[(r+1)A_n \times (r+1)A_n]$ that sum to x. So we have:-

$$|\sum_{n\in F}(2A_n)| \leq \frac{\prod_{n\in F}|(r+1)A_n|^2}{|E_F|}.$$

To complete the second inequality notice that, just as in the first inequality, $|E_F| = \ast(\chi_{rA_n} * \chi_{rA_n})(0)$. □

We now define the following three quantities:-

$$h_1(X) = \sup_A \limsup_{n\to\infty} \left[\frac{-1}{|F_n|}\log\left(\frac{1}{|A|^{2|F_n|}} \ast(\chi_{S^{-1}A} * \chi_{S^{-1}A})(0)\right)\right]$$

where the supremum is taken over all finite and symmetric sets $A \subset \Gamma$.

$$h_2(X) = \sup_\varphi \limsup_{n\to\infty} \left[\frac{-1}{|F_n|}\log\left\{\left(\int\varphi\right)^{-|F_n|} \ast(\varphi S)(0)\right\}\right]$$

where the supremum is taken over all functions $\varphi \in P(\Gamma) \cap C_{00}(\Gamma)^+$ ($C_{00}(\Gamma)^+$ is the space of all non-negative functions on Γ with finite support).

$$h_3(X) = \sup_\varphi \limsup_{n\to\infty} \left\{\frac{-1}{|F_n|}\log\left\{\left(\int\varphi\right)^{-|F_n|} \ast(\varphi S)(0)\right\}\right\}$$

where the supremum is taken over all functions $\varphi \in P(\Gamma) \cap L^1(\Gamma)^+$.

Lemma III.1.5: $h_3(X) = h_{alg}(\Gamma)$

Proof: We divide the proof into three steps. Write $\ast$, $\sum$ and $\prod$ to mean $\ast_{n\in F_n}$, $\sum_{n\in F_n}$ and $\prod_{n\in F_n}$ when the value of n is clear from the context.

<u>Step One</u>: Claim that $h_1(X) = h_{alg}(\Gamma)$.

<u>Proof</u>: Let $A_n = S_{-n}A$ in Lemma III.1.4. From the first inequality of III.1.4, we get:-

$$\frac{-1}{|F_n|}\log\left[|A|^{-2|F_n|} \ast(\chi_{S^{-1}A} * \chi_{S^{-1}A})(0)\right] \le \frac{1}{|F_n|}\log\left|\sum S^{-1}(2A)\right|$$

and so $h_1(X) \le h_{alg}(X)$. From the second inequality of III.1.4:-

$$\frac{1}{|F_n|}\log|\sum S^{-1}(2A)| \le \frac{-1}{|F_n|}\log[\,|rA|^{-2|F_n|} * (\chi_{S^{-1}(rA)} * \chi_{S^{-1}(rA)})(0)\,] + 2\log\frac{|(r+1)A|}{|rA|}$$

Thus we will have shown the claim if $|(r+1)A|/|rA|$ converges to 1 as r goes to infinity. To see this, consider two cases separately.

Torsion free case. Suppose that $A = \{0, \pm a_1, \ldots, \pm a_s\}$ is independent and torsion free. Then $rA = \{\sum_{i=1}^{s} r_i a_i \mid r_i \in \mathbb{Z} \text{ and } \sum_{i=1}^{s}|r_i| \le r\}$ and the limit is clearly 1.

General case[1]. $A \subset \Gamma$ an arbitrary finite subset. Let «A» be the subgroup generated by A, and express «A» as a finite direct sum of cyclic subgroups. Let G be a set containing a generator for each of the cyclic subgroups and write $G = G_{tor} \cup G_\infty$, where G_{tor} is the subset of elements with finite order and G_∞ the subset of elements with infinite order.

Set $p = \prod_{x\in G_{tor}} \mathrm{Order}(x)$, $A_{tor} = G_{tor} \cup -G_{tor} \cup \{0\}$ and $A_\infty = G_\infty \cup -G_\infty \cup \{0\}$.

There is an integer p' such that $A \subset p'(A_{tor} \cup A_\infty)$. Let $q = \max\{p, |p'|\}$ and assume without loss of generality (since entropy is computed over an increasing limit) that $A = q(A_{tor} \cup A_\infty)$.

Now for any $r > q$:-

$$\frac{|(r+1)A|}{|rA|} \le \frac{|(r+1)qA_\infty + pA_{tor}|}{|(r-p)qA_\infty + pA_{tor}|} = \frac{p|(r+1)qA_\infty|}{p|(r-p)qA_\infty|}$$

since $|C + D| = |C| \times |D|$ if C and D are independent. The last expression above converges to 1 by the torsion free case so $h_1(X) \ge h_{alg}(X)$ and therefore $h_1(X) = h_{alg}(X)$. Step one is completed.

[1]This case is included because this section applies to an action on any compact abelian group. For the systems considered in Chapter 2 this problem can be avoided as follows. If R/I has torsion then the generators of I have a non-trivial constant common factor, say s, and it can be shown that $h_{alg}(X_I) = h_{alg}(X_{(I/s)}) + \log s$. The group R/(I/s) is torsion free.

Step Two: Claim that $h_2(X) = h_1(X)$

Proof: If $A \subset \Gamma$ is finite we have $\chi_{S^{-1}A} * \chi_{S^{-1}A} = (\chi_A * \chi_A)S$. So $h_1(X) \leq h_2(X)$ since in computing h_1 we are taking the supremum over the subset of $P(\Gamma) \cap C_{00}(\Gamma)^+$ given by functions of the form $\chi_A * \chi_A$ for A finite and symmetric. Let $\varphi \in P(\Gamma) \cap C_{00}(\Gamma)^+$ have support given by A, and let $\chi_n = \chi_{\sum S^{-1}A}$. Notice that Support$(\ast(\varphi S)) \subset \sum S^{-1}A$ so we have:-

$$\chi_n(\ast(\varphi S))(0) = \int \chi_n(y) * (\ast(\varphi S))(y) d|y| = \int (\ast(\varphi S)) d|y| = \left(\int \varphi\right)^{|F_n|}$$

(Integration is over Γ). So we have the estimate:-

$$1 \leq \left(\int \varphi\right)^{-|F_n|} [\chi_n * (\ast(\varphi S))(0)] = \left(\int \varphi\right)^{-|F_n|} \int \chi_n(y) (\ast(\varphi S))(y) d|y|$$

$$\leq \left(\int \varphi\right)^{-|F_n|} \times \left|\sum_{S \in F_n} S^{-1}A\right| \times (\ast(\varphi S))(0)$$

and hence $h_2(X) \leq h_{alg}(X)$. But we have already shown that $h_{alg}(X) = h_1(X)$ so $h_2(X) \leq h_1(X)$ and step two is completed.

Step three: Claim that $h_3(X) = h_2(X)$

Proof: It is clear from the inclusion $C_{00}(G)^+ \subset L^1(G)^+$ that $h_2(X) \leq h_3(X)$.

Notice that:-

$$\ast((\varphi S) * (\varphi S))(0) = \int [\ast(\varphi S)](y) \times [\ast(\varphi S)](y)\, d|y| \leq [\ast(\varphi S)](0) \times \left(\int \varphi\right)^{|F_n|}$$

so that:-

$$[\int \varphi*\varphi]^{-|F_n|} \times (\ast(\varphi S*\varphi S))(0) \leq [\int \varphi]^{-|F_n|} \times (\ast\varphi S)(0)$$

which gives the alternative description:-

$$h_3(X) = \sup_{\varphi} \limsup_{n\to\infty} [\frac{-1}{|F_n|} \log \{ (\int \varphi*\varphi)^{-|F_n|} \times (\ast(\varphi S*\varphi S))(0)\}]$$

In the definition of h_3 we can assume that φ lies in the unit ball of L^1 so that $\int|\varphi(y)|d|y| = 1$ and $\varphi \in P(\Gamma)$. Given such a φ and any $\varepsilon > 0$, choose $f \in C_{00}(\Gamma)^+$ with $0 \leq f \leq \varphi$, $f(-x) = f(x)^c$ (complex conjugate) and $\|f-\varphi\|_1 < \varepsilon/2$. Then $f * f$ is positive definite, $0 \leq f * f \leq \varphi * \varphi$, and

$$\begin{aligned}\|f * f - \varphi * \varphi\| &\leq \|f * f - f * \varphi\| + \|f * \varphi - \varphi * \varphi\| \\ &\leq \|f\| \times \|f - \varphi\| + \|\varphi\| \times \|f-\varphi\| \\ &< \varepsilon\end{aligned}$$

since $\|f\| \leq \|\varphi\| = 1$ by assumption. Putting $\psi = f * f$ we have $\psi \in P(\Gamma) \cap C_{00}(\Gamma)^+$ and $0 \leq \int\varphi*\varphi - \int\psi \leq \varepsilon$, so:-

$$\frac{-1}{|F_n|} \log [(\int\psi)^{-|F_n|} (\ast(\varphi S))(0)] \geq \frac{-1}{|F_n|} \log [\ast(\varphi S*\varphi S)](0) + \log \int\psi$$

and since $\int\psi \geq (1-\varepsilon)$ we have $h_2(X) = h_3(X)$. This is the third step and we conclude that $h_{alg}(X) = h_3(X)$. □

We now apply the Fourier Inversion Theorem and observe that the expressions h^1 and h_3 in Lemma III.1.3 and III.1.5 for $Bow(X)$ and $h_{alg}(X)$ respectively are dual to each other to conclude that $h_{alg}(\Gamma) = Bow(X)$. This is Theorem III.1.2. □

Corollary III.1.6: $h_{top}(X_M) = h_{alg}(M)$

Proof: This is all shown above; $h_{alg}(M) = Bow(X_M)$ by Theorem III.1.2 and

$\mathrm{Bow}(X_M) = h_{top}(X_M)$ by Lemma II.1.5. □

To illustrate Corollary III.1.6 we compute the algebraic entropy of a very simple casedirectly.

Example III.1.7: Consider the two-dimensional action defined by the constant polynomial s. It is easy to see that the topological entropy is log s directly; there are s^{nm} points of period (n,m) in $X_{<s>}$ so the growth rate of periodic points along a rectangular Følner sequence is log s. To compute the algebraic entropy, write the elements of $\mathbf{R}/\langle s\rangle \cong (\mathbb{Z}/s\mathbb{Z})^{\mathbb{Z}^2}$ in the form of functions $q\colon \mathbb{Z}^2 \longrightarrow \mathbb{Z}/s\mathbb{Z}$. Define the finite cylinder set:

$$E_k = \{\, q \in \mathbf{R}/\langle s\rangle \mid q(i,j) = 0 \ \text{ if } |i| > k \text{ or } |j| > k \,\}.$$

Choose the sequence $F_n = ([-n, n] \cap \mathbb{Z})^2$. Then:

$$F_n(E_k) = \{q \in \mathbf{R}/\langle s\rangle \mid q(i,j)=0 \text{ if } |i|>k+n \text{ or } |j|>k+n\}$$

and so $|F_n(E_k)| = s^{(2(k+n)+1)^2}$. The algebraic entropy for the set E_k is then:

$$h_{alg}(\mathbf{R}/\langle s\rangle, E_k) = \lim_{n\to\infty} \frac{1}{|F_n|} \log |F_n(E_k)| = \lim_{n\to\infty} \frac{1}{(2n+1)^2} \log s^{(2(k+n)+1)^2} = \log s$$

for all k. Thus (since the orbits under the action of E_k eventually covers everything we have an analogue of the Kolmogorov-Sinai theorem) the algebraic entropy is log s.

IV: Zeta Functions

§1: Group actions and ζ functions

In this section we define zeta functions for expansive higher dimensional actions. There is an immediate difficulty because there is only one truly canonical way to define the zeta function and the resulting function turns out to be intractable even in the simplest cases. We will define zeta functions with respect to two different sets of possible periods Ω_r and Ω_m and compare them.

In order to avoid a great deal of complication we state and prove the results for expansive elements of *Id(d)*. This means that the growth rate of periodic points is determined by a convergent sequence along any Følner sequence of periods (see Lind, Schmidt and Ward [1]). Throughout, write h for the global topological entropy.

Given Theorem II.3.5 it would be good to have the smallest real pole of the zeta functions at exp(-h). This will be shown for functions closely related to the zeta functions but does not happen in general. However, the original zeta function does have the property that exp(-h) is a cluster point of the other poles. The other poles correspond to subsystems of lower dimension and arise because there are sequences of periods that are not Følner.

We begin with a one-dimensional example as in Chapter II.

Example IV.1.1: For the case $d = 1$, the zeta function is the usual one for each of the sets Ω_r and Ω_m. That is:-

$$\zeta(s) = \exp \sum_{n=1}^{\infty} |\mathrm{Fix}(n)| \times \frac{s^n}{n}$$

See section I.4 of Smale [1]. Let the ideal defining the system be as in Lemma II.4.2: $p = \langle f \rangle$. By using the calculation there of $|\mathrm{Fix}(n)|$ we deduce that:-

$$\zeta(s) = \prod_{1\le i_1<i_2\ldots<i_k\le n} (1 - \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k} f_n s)^{(-1)^{k+1}}$$

Notice that the least real pole of ζ is at

$$f_n^{-1} \prod_{|\lambda|>1} \lambda_i^{-1}$$

which is at $\exp(-h)$ by Yuzvinskii. This zeta function is rational which does not occur in higher dimensions, despite the obvious finite determinacy of the systems.

Definition IV.1.2: A point $x \in X_p$ is said to be periodic if its stabiliser group H_x has finite index in $\mathbb{Z}^d$. H_x is defined by $\{ n \in \mathbb{Z}^d \mid \alpha_n x = x \}$. A period for x, $\pi(x)$, is any fundamental region for H_x in $\mathbb{Z}^d$. The size of the period, which will be written as $|\pi(x)| = |\mathrm{Orbit}(x)|$ is defined to be $|\mathbb{Z}^d/H_x|$. These ideas were used by the late H. Michel in the study of higher dimensional subshifts of finite type, see Michel [1].

Given a countable collection of periods $\Omega \subset \mathbb{P}(\mathbb{Z}^d)$ each of which tiles the group $\mathbb{Z}^d$ we define the zeta function of the action of $\mathbb{Z}^d$ with respect to the set Ω by:-

$$\zeta_\Omega(t) = \exp \sum_{n=1}^{\infty} |\mathrm{Fix}_\Omega(n)| \times \frac{t^n}{n}$$

where $\mathrm{Fix}_\Omega(n)$ is the set of all periodic points that repeat the pattern defined by an element of Ω whose d-dimensional volume is n.

The two possibilities for the set Ω that we consider are:-

(i) 'Rectangular' zeta functions/periods:-

$$\Omega_r = \{ \prod_{i=1}^{d} [0,n_i] \cap \mathbb{Z}^d \mid n_i \in \mathbb{N} \}$$

(ii) All periods:-

$$\Omega_m = \{ \text{fundamental regions that tile } \mathbb{Z}^d \}$$

Notice that $\Omega_r \subset \Omega_m$, so that for t real and positive, $\zeta_r(t) \le \zeta_m(t)$. Also if $d=1$ then $\Omega_r = \Omega_m$ so that $\zeta_r = \zeta_m$ as mentioned in Example IV.1.1. H. Michel [1] considered zeta functions of type (2) and used them to deduce asymptotic formulae for the distribution of closed orbits in higher-dimensional subshifts of finite type.

We now evaluate ζ_r for the simplest situation - that of the full shift on s symbols in two dimensions, as considered for instance in Example III.1.7.

Example IV.1.3: Let $\mathbf{p} = \langle s\rangle$ and $d = 2$. Then $|\mathrm{Fix}(n, m)| = s^{nm}$. The set of periods in Ω_r with size n is given by the set of pairs (a, b) with $a\cdot b = n$. The number of such pairs is $|\Omega_r(n)| = d(n)$, the number of divisors of n. The rectangular zeta function is then given by:-

$$\zeta_r(t) = \exp \sum_{n=1}^{\infty} \frac{t^n}{n} \times \sum_{|S|=n, S\in\Omega_r} |\{\text{points with period } S\}|$$

$$= \exp \sum_{n=1}^{\infty} d(n) \times \frac{(st)^n}{n}$$

In order to look at this more closely, consider (within the radius of convergence) the associated function $\xi(t) = \zeta'(t)/\zeta(t)$. In our case, $\xi_r(t)$ can be expanded as a *Lambert series* (see Theorem 310 in Hardy and Wright [1]):-

$$\xi_r(t) = \sum_{n=1}^{\infty} d(n)\times(st)^n = \sum_{n=1}^{\infty} (st)^n(1 - (st)^n)^{-1}$$

Remark IV.1.4: Since the number of triples (a, b, c) with $a\cdot b\cdot c = n$, $d_3(n)$, is given by:-

$$d_3(n) = \sum_{k|n} d(k)$$

with similar formulae in higher dimensions, we can deduce that for the full d-dimensional shift on s symbols:-

$$\xi_r(t) = \zeta_r'(t)/\zeta_r(t) = \sum_{n_1,\ldots,n_d=1}^{\infty} (st)^{n_1 \ldots n_d}\left[1 - (st)^{n_1 \ldots n_d}\right]^{-1} \quad \text{if } d \geq 2$$

$$\text{and } \zeta_r(t) = \frac{1}{1-st} \quad \text{if } d = 1$$

In all cases there is a pole at $\exp(-h)$. Any system for which $|\mathrm{Fix}(\mathbf{n})|$ is an exponential function of $(n_1 \times \ldots \times n_d)$ alone will also have $\xi(t)$ expandable as a Lambert series in the same way. For most ideals $\mathbf{p}$ however this does not happen - typically we will have $|\mathrm{Fix}(n, m)| \neq |\mathrm{Fix}(m, n)|$ if $m \neq n$.

When we consider ζ_m there is an immediate problem. The number of elements of Ω_m with given size n does not have any kind of generating function of the kind used above for $d(n)$. There are estimates which show that this number grows slowly (polynomially) and therefore the additional terms in ζ_m will not affect convergence properties. Thus, ζ_m and ζ_r will have the same poles.

Remark IV.1.5: The number of subgroups of $\mathbb{Z}^d$ $(d \geq 2)$ with index n, $|\Omega_m(n)|$, has the estimate:-

$$|\Omega_m(n)| \leq (d-1)\times\left(\frac{d}{2} -1+n\right)\times n^{\log d}\times(d!)$$

See Michel [1]. This shows that for $d > 1$, the zeta function will not be rational even for the full shift on s symbols, since it is easy to show that $\exp\sum a_n(s^n/n)$ is rational if and only if there exist complex numbers α_i, β_i such that:-

$$a_n = \sum_{i=1}^{l}\alpha_i^n - \sum_{j=1}^{k}\beta_j^n$$

Notice that for $d = 1$ we have this relation with $\alpha_1=1$, $l=1$ and $k=0$, where a_n is the number of subgroups of $\mathbb{Z}$ with index n.

§2: Poles and Product Formulæ

In order to prove the weak result about poles of the zeta functions we make the following definition of a restricted function which avoids certain bad periods. In this context, the periods we need to avoid are those that are uniformly flat but arbitrarily large like $\{ [0, 1] \times [0, n] \}_{n \in \mathbb{N}}$: it is possible that some arithmetic weighting (dependent on $d(n)$) could be given to these periods to compensate sufficiently for this.

Definition IV.2.1: Given $\theta > 0$ define the set Ω_θ to be the set of rectangular periods which are not too flat in any variable:-

$$\Omega_\theta = \{[0,n_1] \times \ldots \times [0,n_d] \mid \min\{n_i/n_j \mid i,j=1 \text{ to } d\}>\theta\}$$

Further, let $\zeta_\theta(s) = \zeta_{\Omega_\theta}(s)$ in the notation of Definition IV.1.2 be the corresponding zetafunction.

Lemma IV.2.2: For an expansive system $(X_p, \mathbb{Z}^d)$ (where $V(p)$ has no unit roots), the following holds.

(1) The zeta functions ζ_r, ζ_m and ζ_θ for any $\theta \in (0, 1)$ each have a pole at $\exp(-h_p)$. Within the disk $\{ z \in \mathbb{C} \mid |z| < \exp(-h_p) \}$, ζ_θ is given by an absolutely convergent power series.

(2) If $p \cap \mathbb{Z} = p\mathbb{Z}$ for some $p \neq 0$ then ζ_r and ζ_m are given by absolutely convergent series on the disk $\{ z \in \mathbb{C} \mid |z| < 1/|p| \}$.

(3) If $d = 1$ then ζ_r, ζ_m and ζ_θ for any $\theta \in (0, 1)$ are analytic in the disk $\{ z \in \mathbb{C} \mid |z| < \exp(-h_p) \}$. For $d > 1$ ζ_r and ζ_m may have poles within this disk.

Proof:

(1) To see that each of the zeta functions has a pole at $\exp(-h_p)$, consider a subset of the terms along a Følner sequence and use Theorem II.3.5. Any sequence in Ω_θ is a Følner sequence, for which we have Theorem II.3.5. This means that any limit point of

the sequence a_m (where n^m is a sequence of periods in Ω_θ) defined by:

$$a_m = \left[\mathrm{Fix}(n_1^m, \ldots, n_d^m) \right]^{1/(n_1^m \times \ldots \times n_d^m)}$$

is equal to $\exp(-h_p)$. So the radius of convergence of the series defining ζ_θ is equal to this value.

(2) This follows the result of Michel on higher-dimensional subshifts of finite type exactly. The $d = 1$ case is clear. Consider ζ_m first. We can crudely bound the number of points with period n by:

$$|\mathrm{Fix}(n)| \leq \sum_{[\mathbb{Z}^d:A]=n} |\mathrm{Fix}(A)| \leq \sum_{[\mathbb{Z}^d:A]=n} |p|^n \leq |p|^n (d-1)\times(\frac{d}{2}-1+n)\times n^{\log d}\times(d!)$$

(where we are summing over all the distinct subgroups of index n) from which it is clear that the radius of convergence is bounded by $1/|p|$. By the inclusion $\Omega_r \subset \Omega_m$ it is clear that the same is true of ζ_r.

(3) The $d = 1$ statement is an immediate corollary of Lemma II.4.2. To see that there may be additional poles if $d > 1$, consider the following example. Let $p = \langle 1+x+5y \rangle$. Then we have (Examples II.5.1 *(v)*) $\exp(-h_p) = 1/5$. However, an easy calculation shows that $|\mathrm{Fix}(n,1)| = 6^n-(-1)^n$. So, at $s = 1/6$, the series $\sum |\mathrm{Fix}(n,1)| s^n/n$ diverges. This shows that the series for either of the zeta functions also diverges since all the coefficients are real and positive. Notice that the sequence of periods used here does not lie in Ω_θ for any $\theta > 0$. □

The following lemmas extend the above by explaining how some of the additional poles in ζ_r arise from lower-dimensional entropies of lower-dimensional systems. For the purposes of Lemma IV.2.3 and Corollary IV.2.4, let ζ denote either ζ_m or ζ_r.

Lemma IV.2.3: Given any pair $(\mathbf{m}, j)$ where $\mathbf{m} \in \mathbb{Z}^{d-e}$ with $0 < e < d$ and j is an injective map from $\{1, 2, \ldots, d-e\}$ into $\{1, 2, \ldots, d\}$ there is a well defined $\mathbb{Z}^e$

action $(X_{v(\mathbf{m},j)}, \mathbb{Z}^e)$ where $X_{v(\mathbf{m},j)}$ is the subgroup of points in X with period m_i in the direction $e_{j(i)}$ for $i = 1, \ldots, d-e$. Let the e-dimensional entropy of this subsystem be written $h(X_{v(\mathbf{m},j)}, \mathbb{Z}^e)$. Then there is a pole of the zeta function of $(X, \mathbb{Z}^d)$ at the $|m_1 \times \ldots \times m_{d-e}|^{th}$ root of $h(X_{v(\mathbf{m},j)}, \mathbb{Z}^e)$ for any $(\mathbf{m}, j)$.

Proof: As usual, for s given by a positive real all the terms in the series defining the zeta function are positive so it is sufficient to look along a Følner sequence $\{F_n\}$ in $\mathbb{Z}^e$. Along this sequence the growth rate of periodic points is $h(X_{v(\mathbf{m},j)}, \mathbb{Z}^e)$ and the series has general term: $s^{|F_n| \times |m_1 \times \ldots m_{d-e}|} \times |\mathrm{Fix}(F_n, X_{v(\mathbf{m},j)})|$ from which it is clear that there is a pole at the $|m_1 \times \ldots \times m_{d-e}|$ th root of $h(X_{v(\mathbf{m},j)}, \mathbb{Z}^e)$ for any $(\mathbf{m}, j)$. □

Corollary IV.2.4: Write P for the set of poles of ζ other than $\exp(-h)$. Then if $P \cap \mathbb{R}$ is non-empty[1], $\exp(-h)$ is contained in the closure of $P \cap \mathbb{R}$.

Proof: This follows from the above result and the proof of Theorem II.3.5 where the dimension is reduced by one. A calculation similar to that performed there (identical but notationally hard) will show this; the only possible problem is the possible presence of points in the intersection of $V(\mathbf{p})$ with the d-torus. These do not occur by assumption. □

For the special case[2] $d = 2$, the additional poles described in Lemma IV.2.3 will be of finite order (as in Example IV.1.1). The pole at $\exp(-h)$ is essential so if $P \cap \mathbb{R}$ is not empty the zeta function has a sequence of poles each of finite order converging to a pole of infinite order. This can be seen in the examples below. We combine the above observations in the form of the following lemma. Notice that this lemma has nothing to do with the groups under consideration and is inherent to the rectangular zeta function. It allows the rectangular zeta function to be computed inductively as long as the zeta functions of the one dimensional subsystems can be

[1]The only examples with $P \cap \mathbb{R}$ empty that we have found are the systems determined by constants (the full shifts in higher dimensions) and those with zero entropy, see Lemma III.3.1. We conjecture that these are the only such systems if $d>1$. Even in these cases, there are sequences of complex poles (whose values are of the form $\exp(-h) \times$unit roots) converging to $\exp(-h)$. However, obtaining $\exp(-h)$ as a limit of numbers all with the same modulus is not interesting from a dynamical point of view - it is merely a quirk of the formalism.

[2]For $d>2$ there may be poles of finite order corresponding to one-dimensional subsystems. However, this cannot be guaranteed since the possible one-dimensional entropies may also be entropies of two-dimensional systems, in which case they will appear as essential singularities.

computed.

Lemma IV.2.5: Consider the rectangular[3] zeta function of a system $(X, \mathbb{Z}^d)$. Write $\zeta(t, X_{v(m,j)}, e)$ for the e-dimensional zeta function of the subsystem $(X_{v(m,j)}, \mathbb{Z}^e)$. Then we can describe the zeta function as a product:

$$\zeta(t, X) = \prod_{m\in\mathbb{Z}^{d-e}, 0<e<d;\, j} \zeta(t^{m_1\ldots m_{d-e}}, X_{v(m,j)}, e)$$

Proof: Simply note that we consider all the periods in the expression on the right hand side. Taking logs, the coefficient (write[n] (ξ) for the coefficient of s^n in ξ) of s^n on the right hand side is:

$$\sum_{m,j;\ m_1\ldots m_{d-e}|n} [n/m_1\ldots m_{d-e}]\,(\zeta(s, X_{v(m,j)}) = \sum_{m,j;\ m_1\ldots m_{d-e}|n} |\mathrm{Fix}_{(n/m_1\ldots m_{d-e})}(X_{v(m,j)}) = |\mathrm{Fix}_n(X)|$$

which is the coefficient in the left hand side. □

In order to study these higher-dimensional functions at all systematically relations of a functional type would be needed. We have found none of these, but the following Lemma gives one of several trivial relations. Write $\zeta(s, f, d)$ for the zeta function of $(X_f, \mathbb{Z}^d)$.

Lemma IV.2.6: For either of the functions ζ_r, ζ_m and for any $n \in \mathbb{N}$ we have the relation $\zeta(ns, f, d) = \zeta(s, nf, d)$.

Proof: This follows immediatly from the observation that for any period $A \subset \Omega_m$, we have $|\mathrm{Fix}_A(X_{<nf>}) = n^{|A|} \times |\mathrm{Fix}_A(X_{<f>})$. To see this, notice that the system $(X_{<nf>}, \mathbb{Z}^d)$ is algebraically conjugate to $((\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}^d} \times (X_{<f>}, \mathbb{Z}^d))$. □

[3] A similar property holds for ζ_m. The zeta function coincides with the product over suitably weighted zeta functions for all the lower dimensional systems defined by a periodicity in some lattice.

Computing some non-trivial examples of zeta functions in closed form would also help. Lemma IV.2.3 shows that the zeta functions must be fairly complicated except in the exceptional case where all the subsystem entropies coincide. The following examples are highly restricted in that the defining polynomial involves either only one of the variables or is a constant. They exhibit some of the properties described in Lemma IV.2.2, Lemma IV.2.3 and Corollary IV.2.4. Example *(i)* has poles at the points $\lambda_{n,k} = \dfrac{e^{2\pi ik/n}}{\sqrt[n]{3^n+(-1)^n}}$ for $k=1$ to n and any $n \in \mathbb{N}$. The subset of poles determined by $k = 0$ converge to the value $1/3$, which is $\exp(-h)$. Example *(ii)* has the property that the subsystems are all products of k-shifts; thus all the possible weighted entropies coincide and the set $P \cap \mathbb{R}$ is empty. There are poles at $(e^{2\pi ij/n})/k$ for all n and j. Example *(iii)* illustrates a relation between the zeta functions of the full k shift in various dimensions[4]. This is a special case of Lemma IV.2.5 above.

Examples IV.2.7:

(i) $\zeta_r(s, 3-x, 2)=\displaystyle\prod_{n=1}^{\infty}[1-s^n(3^n+(-1)^n)]^{-1/n} = \frac{1}{(1-2s)^1}\times\frac{1}{(1-10s^2)^{1/2}}\times\frac{1}{(1-26s^3)^{1/3}}\times\dots$

(ii) $\zeta_r(s, k, 2) = \dfrac{1}{(1-ks)^1}\times\dfrac{1}{(1-k^2s^2)^{1/2}}\times\dfrac{1}{(1-k^3s^3)^{1/3}}\times\dots$

(iii) $\zeta(s, k, 2) = \displaystyle\prod_{n=1}^{\infty}\zeta(s^n, k^n, 1)^{1/n}$ and $\zeta(s, k, 3) = \displaystyle\prod_{n=1}^{\infty}\zeta(s^n, k^n, 2)^{1/n}$ with similar formulae in higher dimensions.

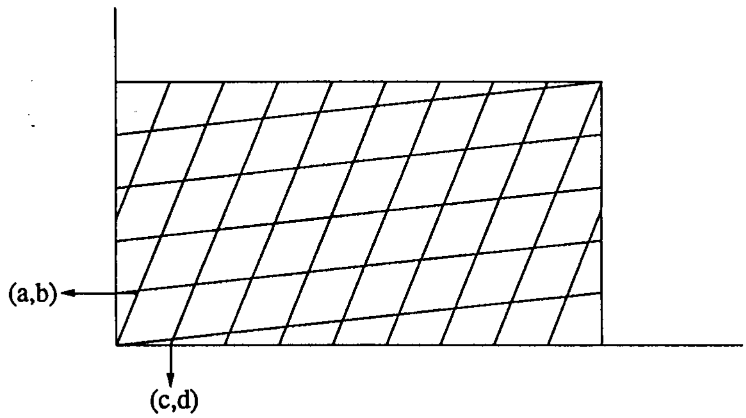
We now turn to the problem of product formulae.

Definition IV.2.8: Let Θ be the set of all finite orbits in X_I under the action of $\mathbb{Z}^d$. Define the maps $r, |\,| : \Theta \longrightarrow \mathbb{N}$ as follows where $\tau \in \Theta$:-

[4]For $d=1$ it is easy to see that $\zeta(s,k,1) = 1/(1-ks)$.

(1) $r(\tau)$ = the size of the least rectangular period satisfied by an $x \in \tau$.
(2) $|\tau|$ = volume of the periodicity of $\tau = |\mathbb{Z}^d/H_x|$ for $x \in \tau$.

Before looking at the product formulae, we look at the $d = 2$ case in more detail and show how the map r works. A given $\tau \in \Theta$ can be described as a 2×2 matrix $[\tau]$ with columns $(a, b)^t$ and $(c, d)^t$ where without loss of generality b and d are non-negative. Let $L[\tau]$ be the lattice generated by (a, b) and (c, d) in $\mathbb{Z}^2$. Notice that $L[\tau] = H_x$ for any $x \in \tau$. The map r can be seen in this diagram:-



In this case, we would have $r(\tau) = 45 \times |\tau|$.

To see that r is well defined we show that any periodic point must have a square period and *a fortiori* a rectangular one. For this we need the following Lemma.

Lemma IV.2.9: For any $\tau \in \Theta$, $L[\tau] \cap \{ (n, n) \mid n \in \mathbb{Z}\backslash\{0\} \} \neq \emptyset$.

Proof: The matrix $[\tau]$ is nonsingular since the determinant is equal to $|\tau| \neq 0$. Therefore:-

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}\begin{bmatrix} n \\ n \end{bmatrix}$$

solves the equation $A(a,b) + B(c,d) = (n,n)$ over $\mathbb{Q}$. Clearing denominators shows there is a point in the intersection. This point defines a square period for any $x \in \tau$. □

It is clear that $r(\tau) = |(pa-qc) \times (rd-sb)|$ where $p/q = d/b, r/s = a/c$ in their lowest terms.

Lemma IV.2.9 cannot be strengthened very much because there is no uniform relationship between the quantities $|\tau|$ and $r(\tau)$. In order to see this, consider the period defined by the shape with corners at (0, 0), (1, 1), (n, 0), and (n, 1). A point whose orbit τ has this period as its least periodicity would have $r(\tau) = |\tau| \times n$. So there is the uniform lower bound $1 \leq r(\tau)/|\tau|$ but there is no possible upper bound.

Lemma IV.2.10: The functions ζ_m and ζ_r satisfy an Euler-Lagrange product formula.

Proof: The relation for ζ_m is due to Michel:-

$$\zeta_m(t) = \prod_{\tau\in\Theta}(1 - t^{|\tau|})^{-1}$$

We claim that the rectangular zeta function has:-

$$\zeta_r(t) = \prod_{\tau\in\Theta}(1 - t^{r(\tau)})^{-1}$$

To see this, let K_n be the number of points with least rectangular period n, and let $F_n=|\mathrm{Fix}_{\Omega_r}(n)|$. Then we have:-

$$F_n = \sum_{m|n} K_m$$

Now we have:-

$$\prod_{\tau\in\Theta}(1 - t^{r(\tau)})^{-1} = \exp\sum_{\tau\in\Theta}\sum_{k=1}^{\infty}\frac{t^{r(\tau)k}}{k}$$

The coefficient of t^n is:-

$$\sum_{k|n} \frac{1}{k} \times |\{\tau \in \Theta \text{ with } r(\tau) = n/k\}|$$

Notice that $|\{\ \tau \in \Theta$ with $r(\tau) = n\ \}| = |\{\ x \in X$ with least rectangular period $n\ \}|/n$. So we have:-

$$\sum_{k|n} \frac{1}{k} \times |\{\tau \in \Theta \text{ with } r(\tau) = n/k\}| = \sum_{k|n} \frac{1}{k} \times K_{(n/k)} \times (k/n)$$

$$= \sum_{k|n} \frac{1}{n} \times K_{(n/k)}$$

$$= \frac{1}{n} \sum_{l|n} K_l = F_n/n$$

and therefore the rectangular product formula. □

§3: Poles of the zeta function and the Mahler measure

In this section we sketch some properties of the poles of the rectangular zeta function. Throughout, let $(X, \mathbb{Z}^d)$ denote the system determined by the polynomial f, let h be the global topological entropy of this system and write $\zeta(s, f, d)$ for the rectangular zeta function. For subsystems and their entropies follow the notation of Section 2 above. We first show that the family of systems with zero entropy has the property that the set $P \cap \mathbb{R}$ is empty.

Lemma IV.3.1: If h is zero and the system is determined by a principal ideal[5], then the only poles of the zeta function lie on the unit circle.

[5]This is of course not true if we allow non-principal ideals. For instance, the ideal $\langle 2, x-y \rangle$ determines a two-dimensional zero entropy system with one-dimensional subsystems whose entropy is log 2.

Proof: This follows from the result mentioned in Examples II.5.1 *(iv).* Writing f in standard form, we know (see Boyd [2], Smyth [1]) that h is zero if and only if f is a product of cyclotomic polynomials in monomials. The essential singularity of ζ must lie at 1. To find the others, notice that any subsystem of the form $(X_{v(m,j)}, \mathbb{Z}^e)$ has the same entropy as a system determined by a polynomial that is a product of powers of cyclotomics. This follows from the proof of Theorem II.4.5. Thus, the e-dimensional entropy of the subsystem is going to be zero, and so all the poles of ζ must be unit roots.□

Now assume that we are in the opposite (usual) situation: the set $P \cap \mathbb{R}$ is non-empty. We will state a result conjectured by David Boyd [2], proved by Wayne Lawton [2] and used by Doug Lind and Klaus Schmidt to derive the formula for the topological entropy given in Theorem II.1.9. Then we will explain the connection between the result and the Følner sequences of Chapter II. Given $\mathbf{r} \in \mathbb{Z}^d$ write $q(\mathbf{r}) = \min\{|\mathbf{n}|_\infty \mid \mathbf{n} \neq 0 \text{ and } \mathbf{n}\cdot\mathbf{r} = 0\}$ and let $f_\mathbf{r}(u) = f(u^{r_1}, \ldots, u^{r_d})$. Write $M(f)$ for the Mahler measure of f so that $M(f) = \exp(h)$.

Proposition IV.3.2: For any polynomial f we have

$$\lim_{q(\mathbf{r})\to\infty} M(f_\mathbf{r}) = M(f)$$

Call this convergence (a). Using this result we can strengthen Corollary IV.2.4 above: in the case $P \cap \mathbb{R}$ non-empty, there is a sequence of one dimensional subsystems whose corresponding (finite degree) poles converge to the essential singularity.

In order to relate this to Chapter II, we follow Lind, Schmidt and Ward [1] and choose the following sequence. Let $\mathbf{r}_n = (1, n, n^2, \ldots, n^{d-1})$. Then $q(\mathbf{r}_n) = n$ so this is a suitable limiting sequence of periods, and we have $M(f_{\mathbf{r}_n}) \longrightarrow M(f)$. Now from Chapter II we know that if $\mathbf{m}$ is a Følner sequence in $\mathbb{Z}^{d-1}$ then

$$h_{top} = \lim_{\mathbf{m}\to\infty} \frac{1}{m_2 \ldots m_d} h_{top}(X_{f,m_2e_2, \ldots, m_de_d}, \mathbb{Z})$$

Call this convergence (b). Now look at $X_{<f_{r_n}>}$ as a one-dimensional system. We have $f_{r_n}(u) = (u, u^n, u^{n^2}, \ldots, u^{n^{d-1}})$. Consider the Følner sequence of periods $[0, n] \times [0, n^2] \times \ldots \times [0, n^{d-1}]$. The system $X_{f, ne_2, \ldots, n^{d-1}e_d}$ has the same entropy as the system given by X_{f_n} where f_n is given by:

$$f_n(u) = \prod_{j_1=0..n-1} \prod_{j_2=0..n^2-1} \cdots \prod_{j_{d-1}=0..n^{d-1}-1} f(u, e^{2\pi i j_1/n}, e^{2\pi i j_2/n^2}, \ldots, e^{2\pi i j_{d-1}/n^{d-1}})$$

Now compare f_n and f_{r_n}: they have the same degree in each variable and have the same set of zeros, so that up to association and again omitting the details, they are identical. Thus[6] the convergence (a) for this sequence is identical to the convergence (b).

[6]The impression that we may be gaining something for nothing, i.e. showing IV.3.2 from the results in Chapter II, is of course false. The deep result IV.3.2 is needed to derive the entropy formula given in Lind, Schmidt and Ward [1]. The results of Chapter II have the same logical status as Lemma IV.3.2: quantities are shown to be equal by observing that their formulae coincide.

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Automorphisms of solenoids and p-adic entropy*

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Abstract. We show that a full solenoid is locally the product of a euclidean component and p-adic components for each rational prime p. An automorphism of a solenoid preserves these components, and its topological entropy is shown to be the sum of the euclidean and p-adic contributions. The p-adic entropy of the corresponding rational matrix is computed using its p-adic eigenvalues, and this is used to recover Yuzvinskii's calculation of entropy for solenoidal automorphisms. The proofs apply Bowen's investigation of entropy for uniformly continuous transformations to linear maps over the adele ring of the rationals.

1. *Background and results*

A *solenoid* is a finite-dimensional, connected, compact abelian group. Equivalently, its dual group is a finite rank, torsion-free, discrete abelian group, i.e. a subgroup of $\mathbb{Q}^d$ for some $d \geq 1$. Solenoids generalize the familiar torus groups. Halmos [**H**] first observed that (continuous) automorphisms of compact groups must preserve Haar measure, providing an interesting class of examples for ergodic theory. Furthermore, Berg [**Be**] has shown that the entropy of such an automorphism with respect to Haar measure coincides with its topological entropy.

We are concerned here with the computation of the topological entropy of an automorphism of a solenoid. If A is such an automorphism, its dual automorphism extends to an element of $GL(d, \mathbb{Q})$, which we also call A (see § 3). When A is a toral automorphism, so $A \in GL(d, \mathbb{Z})$, then the topological entropy of A is given by the familiar formula

$$h(A) = \sum_{|\lambda_i|>1} \log |\lambda_i|, \tag{1}$$

where A has complex eigenvalues $\lambda_1, \ldots, \lambda_d$ counted with multiplicity. To state the generalization to solenoids, let $\chi_A(t)$ be the characteristic polynomial of $A \in GL(d, \mathbb{Q})$, and s denote the least common multiple of the denominators of the coefficients of $\chi_A(t)$. Yuzvinskii [**Y**] proved that

$$h(A) = \log s + \sum_{|\lambda_i|>1} \log |\lambda_i|. \tag{2}$$

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Our purpose here is to explain Yuzvinskii's calculation in terms of a combination of geometric and arithmetic hyperbolicity. We begin in § 3 by lifting A to an automorphism of the full solenoid $\Sigma^d \cong \hat{\mathbb{Q}}^d$ with the same entropy. In Lemma 4.1 we show that the full solenoid is locally a product of a euclidean component and p-adic components for each rational prime p. The entropy of A is computed in Theorem 1 to be the sum of a contribution from the euclidean component, generated by geometric expansion, and contributions from each of the p-adic components, generated by arithmetic expansions. If $\mathbb{Q}_p$ denotes the p-adic completion of the rationals, than a p-adic component contributes the Bowen entropy $h(A; \mathbb{Q}_p^d)$ of the uniformly continuous linear map A on the non-compact metric space $\mathbb{Q}_p^d$. Since the infinite place ∞ on $\mathbb{Q}$ gives the completion $\mathbb{Q}_\infty = \mathbb{R}$, and the Bowen entropy of a linear map on $\mathbb{R}^d$ is given by (1), we can summarize Theorem 1 by

$$h(A) = \sum_{p \leq \infty} h(A; \mathbb{Q}_p^d), \tag{3}$$

i.e. the entropy of the solenoidal automorphism is the sum, over all inequivalent completions of $\mathbb{Q}$, of the entropies of the corresponding linear maps.

In Theorem 2 the p-adic entropy of A is explicitly computed as

$$h(A; \mathbb{Q}_p^d) = \sum_{|\lambda_j^{(p)}|>1} \log |\lambda_j^{(p)}|_p, \tag{4}$$

where the $\lambda_j^{(p)}$ are the p-adic eigenvalues of A lying in a finite extension of $\mathbb{Q}_p$, and $|\cdot|_p$ is normalized so $|p|_p = p^{-1}$. As pointed out to us by the referee, this shows that $h(A; \Sigma^d)$ is the sum of the logarithmic heights of the eigenvalues of A in the sense of algebraic geometry (see [Lan], p. 52). Using (4), we show in Theorem 3 that

$$\sum_{p<\infty} h(A; \mathbb{Q}_p^d) = \log s,$$

so that the mysterious initial term in Yuzvinskii's formula (2) is just the sum of the p-adic entropies of A over $p < \infty$, while the second term is the euclidean term corresponding to $p = \infty$.

The calculation of entropy for group automorphisms has a history going back to the original papers defining entropy. Sinai [S, 1959] showed (1) for 2-dimensional toral automorphisms, and claimed the higher dimensional formula holds. The 2-dimensional case was reproved by Rohlin [R, 1961] as an application of his measurable partition machinery. Abromov [Ab, 1959] computed entropy for automorphisms of 1-dimensional solenoids. Here the map is specified by a rational number m/n in lowest terms and then

$$h([m/n]) = \max \{\log |m|, \log |n|\}. \tag{5}$$

In [G, 1961], Genis claimed without proof the formula (1) for general toral automorphisms. Next Arov [A, 1964] published a proof of (1), and generalized to solenoidal automorphisms whose characteristic polynomial has coefficients whose denominators are all powers of a fixed integer. Finally, Yuzvinskii [Y, 1967] obtained (2) in full generality, using rather complicated linear algebra.

In § 2 we give some examples to illustrate the interplay between geometric and arithmetic components, and describe a combinatorial formulation of entropy due to Peters. Reduction to full solenoids is carried out in § 3. Our approach in § 4 to the proof of the main formula (3) is to realize the full d-dimensional solenoid as the quotient of the adele ring $\mathbb{Q}_{\mathbb{A}}^d$ by the embedded lattice $\mathbb{Q}^d$. Then A lifts to a linear map of $\mathbb{Q}_{\mathbb{A}}^d$ preserving entropy, and a series of reductions shows $h(A; \mathbb{Q}_{\mathbb{A}}^d)$ equals the right side of (3). A technical problem is that Bowen's definition of entropy is in general not additive over products, so some care is needed. In § 5 we deduce the eigenvalue expression (4) for p-adic entropy, and this is used in § 6 to recover Yuzvinskii's formula (2). We remark that analogous results hold if $\mathbb{Q}$ is replaced by a finite algebraic extension k of $\mathbb{Q}$, and $A \in GL(d, k)$. The details of this generalization are contained in [W, § 4].

The possibility of computing entropy for solenoidal automorphisms using p-adic eigenvalues was described without proof in a lecture by the first author [L1], and ultimately goes back to a suggestion of H. Furstenberg. The authors discussed this topic during the 1986 Warwick Symposium, leading the second author to discover the appropriate framework and proofs [W]. The second author expresses his thanks to Klaus Schmidt for his help and advice.

2. *Examples and a combinatorial interpretation*

Before beginning the proof, we give some examples of (3) to illustrate the interplay between the geometric and arithmetic contributions. We also give an algebraic or combinatorial interpretation of entropy due to Peters.

First consider $A = [3/2]$ acting on the 1-dimensional solenoid $\Sigma = \hat{\mathbb{Q}}$. Abromov's result (5) shows $h(A; \Sigma) = \log 3$. Now (3) shows

$$h(A; \Sigma) = h(A; \mathbb{Q}_2) + h(A; \mathbb{Q}_3) + h(A; \mathbb{R}) \tag{6}$$

since by (4) the other components vanish. If $\log^+ x$ denotes $\max\{\log x, 0\}$, then by (4) and (1) we see $h(A; \mathbb{Q}_2) = \log^+ |3/2|_2 = \log 2$, $h(A; \mathbb{Q}_3) = \log^+ |3/2|_3 = 0$, and $h(A; \mathbb{R}) = \log 3/2$, combining to give $h(A; \Sigma) = \log 3$. Here there are positive contributions from the euclidean and 2-adic components. Next consider $A^{-1} = [2/3]$. Then (6) holds with A replaced by A^{-1}. Now (4) shows

$$h(A^{-1}; \mathbb{Q}_2) = \log^+ |2/3|_2 = 0, \qquad h(A^{-1}; \mathbb{Q}_3) = \log^+ |2/3|_3 = \log 3,$$

and $h(A^{-1}; \mathbb{R}) = 0$, again combining to give $h(A^{-1}, \Sigma) = \log 3$. Note, however, that here the euclidean and 2-adic components contribute nothing, and that all entropy comes from the 3-adic direction.

Next consider

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 6/5 \end{bmatrix}$$

acting on the 2-dimensional solenoid Σ^2. Here $\chi_A(t) = t^2 - \frac{6}{5}t + 1$. The complex eigenvalues of A have modulus 1, so $h(A; \mathbb{R}^2) = 0$. Since $p = 5$ is the only arithmetic component with a positive contribution, we see by (3) that $h(A; \Sigma^2) = h(A; \mathbb{Q}_5^2)$. Over $\mathbb{Q}_5$ we have $\chi_A(t) = (t - \lambda_1)(t - \lambda_2)$, where $|\lambda_1|_5 = 5$ and $|\lambda_2|_5 = 5^{-1}$. Thus by (4) we have $h(A; \Sigma^2) = \log 5$, and of course $h(A^{-1}; \Sigma^2) = \log 5$ by the same calculation. Here A is an isometry on the geometric component, while all hyperbolic behavior is concentrated on the 5-adic component. This example was given in [L2] to show that exponential recurrence for solenoidal automorphisms can be entirely due to arithmetic hyperbolicity.

There is an algebraic way to compute entropy using the growth of sums of images of a finite set in the dual group, due to Peters [P]. Let Γ be a discrete abelian group, and A be an automorphism of Γ. For a finite set $E\subset\Gamma$, let $|E|$ denote its cardinality. Put

$$h_{\mathrm{alg}}(A;E)=\limsup_{n\to\infty}\frac{1}{n}\log|E+A^{-1}E+\cdots+A^{-(n-1)}E|,$$

and

$$h_{\mathrm{alg}}(A;\Gamma)=\sup\{h(A;E)\colon E\subset\Gamma,\ E\text{ finite}\}. \tag{7}$$

Peters showed that $h_{\mathrm{alg}}(A;\Gamma)$ coincides with the topological entropy $h(A,\hat{\Gamma})$ of the dual automorphism.

In our case, $\Gamma=\mathbb{Q}^d$ and $A\in GL(d,\mathbb{Q})$. Then (3) and (4) can be used to compute $h_{\mathrm{alg}}(A;\mathbb{Q}^d)$. The reader may find it instructive to prove directly that $h_{\mathrm{alg}}([3/2];\mathbb{Q})=\log 3$.

3. *Full solenoids*

In this section we lift an automorphism of a solenoid to one of a full solenoid while preserving entropy. This allows us to assume from now on that G is a full d-dimensional solenoid Σ^d, i.e. the dual of G is $\mathbb{Q}^d$.

Let G be a solenoid, and Γ its dual group. Since Γ has finite rank, say d, and is torsion-free, it embeds in $\Gamma_{\mathbb{Q}}=\Gamma\otimes_{\mathbb{Z}}\mathbb{Q}\cong\mathbb{Q}^d$. Hence an automorphism A of Γ extends to an automorphism $A_{\mathbb{Q}}$ of $\Gamma_{\mathbb{Q}}$. Let $G_{\mathbb{Q}}=\hat{\Gamma}_{\mathbb{Q}}$, and $A_{\mathbb{Q}}$ also denote the dual automorphism. By duality, A is a quotient of $A_{\mathbb{Q}}$, and we claim entropy is preserved. Note that $G_{\mathbb{Q}}\cong\Sigma^d$ is a full solenoid, and that $A_{\mathbb{Q}}$ can be considered a rational matrix in $GL(d,\mathbb{Q})$.

PROPOSITION 3.1. *With the above notations,* $h(A;G)=h(A_{\mathbb{Q}};G_{\mathbb{Q}})$.

Proof. For $n\geq 1$ the subgroups $\Gamma_n=(n!)^{-1}\Gamma$ of $\Gamma_{\mathbb{Q}}$ are A-invariant, and increase to $\Gamma_{\mathbb{Q}}$. Hence $G_{\mathbb{Q}}$ is the inverse limit of the A-invariant $\hat{\Gamma}_n$, and

$$h(A_{\mathbb{Q}};G_{\mathbb{Q}})=\lim_{n\to\infty}h(A_{\mathbb{Q}};\hat{\Gamma}_n).$$

Since Γ is torsion-free, the action of $A_{\mathbb{Q}}$ on Γ_n is isomorphic to its action on Γ, so $h(A_{\mathbb{Q}};\hat{\Gamma}_n)=h(A;G)$ for $n\geq 1$. This proves the result. □

Although entropy is preserved when lifting A to $A_{\mathbb{Q}}$, other dynamical properties may be lost. For example, consider $\Gamma=\mathbb{Z}[1/6]$ and $A=[3/2]$, and put $G=\hat{\Gamma}$. The closure of the subgroup of A-periodic points in G has annihilator

$$\bigcap_{n=1}^{\infty}\left[\left(\frac{3}{2}\right)^n-1\right]\Gamma=\{0\},$$

i.e. the periodic points of A in G are dense. However, passing to $\Gamma_{\mathbb{Q}}\cong\mathbb{Q}$, and noting that

$$\bigcap_{n=1}^{\infty}\left[\left(\frac{3}{2}\right)^n-1\right]\Gamma_{\mathbb{Q}}=\Gamma_{\mathbb{Q}},$$

we see that $A_{\mathbb{Q}}$ has only 0 as a periodic point. Here A may be thought of as hyperbolic with eigendirections being the p-adic components with $p=2,3$, and ∞. Passing to $A_{\mathbb{Q}}$ introduces isometric directions for the other primes that destroy periodic behavior.

For a general solenoidal automorphism A it is not difficult to show that its periodic points are dense precisely when it is expansive, and this occurs if and only if the dual group is finitely generated as a $\mathbb{Z}[A, A^{-1}]$-module [**La**].

4. *Proof of the main formula*

In this section we prove the main formula (3). Let $A \in GL(d, \mathbb{Q})$. By duality, A acts as an automorphism of the d-dimensional solenoid Σ^d, and also as a uniformly continuous linear map on $\mathbb{Q}_p^d$ for $p \le \infty$. The relation between the entropies of these actions is as follows.

THEOREM 1. $h(A; \Sigma^d) = \sum_{p \le \infty} h(A; \mathbb{Q}_p^d)$.

A convenient approach is the use of the adele ring $\mathbb{Q}_{\mathbb{A}}$ of $\mathbb{Q}$. We will use notations and results from Weil's elegant book [**We**].

For each rational prime p, let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$ with respect to the p-adic valuation $|\cdot|_p$, normalized so $|p|_p = p^{-1}$. As is standard, $p=\infty$ corresponds to the usual absolute value $|\cdot|_\infty$ on $\mathbb{Q}$, so $\mathbb{Q}_\infty \cong \mathbb{R}$. The valuations $|\cdot|_p$ for $p \le \infty$ form a complete list of mutually inequivalent valuations on $\mathbb{Q}$. The phrase "almost every p" will mean "all but a finite number of p." Define the *adele ring* $\mathbb{Q}_{\mathbb{A}}$ of $\mathbb{Q}$ by

$$\mathbb{Q}_{\mathbb{A}} = \left\{ x \in \prod_{p \le \infty} \mathbb{Q}_p : |x_p|_p \le 1 \qquad \text{for almost every } p \right\}.$$

For a finite set $P \subset \{2, 3, 5, \ldots\} \cup \{\infty\}$ with $\infty \in P$, put

$$\mathbb{Q}_{\mathbb{A}}(P) = \{x \in \mathbb{Q}_{\mathbb{A}} : |x_p|_p \le 1 \qquad \text{if } p \notin P\}.$$

Each $\mathbb{Q}_{\mathbb{A}}(P)$ is locally compact under the product topology, and the topology on $\mathbb{Q}_{\mathbb{A}}$ is the coarsest making each of the $\mathbb{Q}_{\mathbb{A}}(P)$ an open subring. Under this topology $\mathbb{Q}_{\mathbb{A}}$ itself is locally compact. For $x \in \mathbb{Q}$, let $\delta(x) \in \mathbb{Q}_{\mathbb{A}}$ be the diagonal embedding given by $\delta(x)_p = x$ for $p \le \infty$.

LEMMA 4.1. *The subgroup $\delta(\mathbb{Q})$ is discrete in $\mathbb{Q}_{\mathbb{A}}$, and $\mathbb{Q}_{\mathbb{A}}/\delta(\mathbb{Q}) \cong \Sigma$.*

Proof. See Theorems 2 and 3 of [**We**, § IV.2]. □

Identifying $\mathbb{Q}$ with $\delta(\mathbb{Q}) \subset \mathbb{Q}_{\mathbb{A}}$, we may consider $\mathbb{Q}_{\mathbb{A}}$ as a rational vector space. Therefore the action of A on $\mathbb{Q}^d$ extends to $\mathbb{Q}_{\mathbb{A}}^d$ by definining $(Ax)_p = A(x_p)$ for $x \in \mathbb{Q}_{\mathbb{A}}^d$. Using the identifications of Theorem 3 of [**We**, § IV.2], the quotient action of A on $\mathbb{Q}_{\mathbb{A}}^d/\delta(\mathbb{Q})^d$ is isomorphic to that of A on Σ^d. Since $\mathbb{Q}_{\mathbb{A}}^d$ is locally compact metric, and A is uniformly continuous, the definition of topological entropy $h(A; \mathbb{Q}_{\mathbb{A}}^d)$ of Bowen [**B**] applies.

LEMMA 4.2. $h(A; \Sigma^d) = h(A; \mathbb{Q}_{\mathbb{A}}^d)$.

Proof. By the above, $h(A; \Sigma^d) = h(A; \mathbb{Q}_{\mathbb{A}}^d/\delta(\mathbb{Q})^d)$. Since $\delta(\mathbb{Q})^d$ is a discrete subgroup of $\mathbb{Q}_{\mathbb{A}}^d$ with compact quotient, the result follows from [**B**, Thm. 20]. □

Our method to compute topological entropy uses Haar measure to count orbits, following Bowen [B]. Suppose (X, ρ) is a locally compact metric space, that $T: X \to X$ is uniformly continuous, and put

$$D_n(x, \varepsilon, T) = \bigcap_{k=0}^{n-1} T^{-k}(B_\varepsilon(T^k x)),$$

where $B_\varepsilon(y) = \{z \in X : \rho(y, z) < \varepsilon\}$. A Borel measure μ on X is called *T-homogeneous* [B, Def. 6] if

(1) $\mu(K) < \infty$ for all compact K,

(2) $\mu(K) > 0$ for some compact K,

(3) for each $\varepsilon > 0$ there are $\delta > 0$ and $c > 0$ so that

$$\mu(D_n(y, \delta, T)) \le c\mu(D_n(x, \varepsilon, T))$$

for all $n \ge 0$ and $x, y \in X$.

If μ is T-homogeneous, put

$$k(\mu, T) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu(D_n(x, \varepsilon, T)).$$

By condition (3), this is independent of x. The key result from Bowen [B, Prop. 7] is that $k(\mu, T) = h(T)$ for any T-homogeneous measure μ. In particular, if A is an automorphism of a locally compact group G, then Haar measure μ_G is A-homogeneous, so $h(A) = k(\mu_G, A)$.

To apply this method, we find a finite set of primes for A which are those contributing to its entropy. If $\mathbb{Z}_p$ denotes the p-adic integers, then both A and A^{-1} have entries in $\mathbb{Z}_p$ for almost every p. Let P be the set of primes p for which some entry of A or of A^{-1} is not in $\mathbb{Z}_p$, together with ∞. Thus $A \in GL(d, \mathbb{Z}_p)$ for $p \notin P$.

LEMMA 4.3. $h(A; \mathbb{Q}_{\mathbb{A}}^d) = h(A; \mathbb{Q}_{\mathbb{A}}(P)^d)$.

Proof. Since $A \in GL(d, \mathbb{Z}_p)$ for $p \notin P$, it follows that $\mathbb{Q}_{\mathbb{A}}(P)^d$ is an A-invariant neighborhood of the identity. Since Harr measure on $\mathbb{Q}_{\mathbb{A}}$, which is the restriction of Haar measure on $\mathbb{Q}_{\mathbb{A}}(P)$, is A-homogeneous, it follows from [B, Prop. 7] that $h(A; \mathbb{Q}_{\mathbb{A}}^d) = h(A; \mathbb{Q}_{\mathbb{A}}(P)^d)$. □

LEMMA 4.4. $h(A; \mathbb{Q}_{\mathbb{A}}(P)^d) = h(A; \prod_{p \in P} \mathbb{Q}_p^d)$.

Proof. Since $\mathbb{Q}_{\mathbb{A}}(P)^d = \prod_{p \in P} \mathbb{Q}_p^d \times \prod_{p \notin P} \mathbb{Z}_p^d$, with the second factor compact, it follows from [Wa, Thm. 7.10] that

$$h(A; \mathbb{Q}_{\mathbb{A}}(P)^d) = h\left(A; \prod_{p \in P} \mathbb{Q}_p^d\right) + h\left(A; \prod_{p \notin P} \mathbb{Z}_p^d\right).$$

If F is any finite set of primes in P^c, and $m > 0$, then

$$\prod_{p \in F} p^m \mathbb{Z}_p^d \times \prod_{p \in P^c \setminus F} \mathbb{Z}_p^d$$

is an A-invariant neighborhood of 0 since $A \in GL(d, \mathbb{Z}_p)$ for $p \in P^c$. Such neighborhoods form a basis, and again using [B, Prop. 7] with the A-homogeneous Haar measure, we find $h(A; \prod_{p \notin P} \mathbb{Z}_p^d) = 0$. □

LEMMA 4.5. $h(A; \prod_{p \in P} \mathbb{Q}_p^d) = \sum_{p \in P} h(A; \mathbb{Q}_p^d)$.

Proof. An argument is required, since Bowen's definition of entropy is not in general additive over products [**Wa**, p. 176]. Let μ_p be Haar measure on $\mathbb{Q}_p$, and $\mu = \prod_{p\in P} \mu_p$. Then A is μ_p^d-homogeneous. We will show in the proof of Theorem 2 that if B is a compact open subring in $\mathbb{Z}_p$, then

$$-\frac{1}{n}\log \mu_p^d\left(\bigcap_{k=0}^{n-1} A^{-k}(B^d)\right) \to h(A;\mathbb{Q}_p^d) \qquad \text{as} \quad n\to\infty,$$

the essential point being that in this case lim sup's are actually limits. Additivity of entropy then follows. □

Proof of Theorem 1. From Lemmas 4.2–4.5, we conclude that

$$h(A;\Sigma^d) = \sum_{p\in P} h(A;\mathbb{Q}_p^d).$$

An argument as in the proof of Lemma 4.4 shows $h(A;\mathbb{Q}_p^d)=0$ for $p\notin P$. This completes the proof. □

5. *Calculation of p-adic entropy*

It remains to compute the p-adic entropy of the action of A on $\mathbb{Q}_p^d$. The formula and arguments are similar to the euclidean case, with p-adic eigenvalues replacing complex ones.

THEOREM 2. *If A has p-adic eigenvalues $\lambda_1,\ldots,\lambda_d$ in some finite extension of $\mathbb{Q}_p$, then*

$$h(A;\mathbb{Q}_p^d) = \sum_{|\lambda_j|_p>1} \log|\lambda_j|_p,$$

where eigenvalues are counted with multiplicity.

Proof. Let K be a finite extension of $\mathbb{Q}_p$ containing all roots of $\chi_A(t)$, and set $r=[K:\mathbb{Q}_p]$. Then $\mathbb{Q}_p^d\otimes_{\mathbb{Q}_p} K \cong K^d$, and A extends to $A\otimes 1_K$ acting on K^d. Since K is a vector space of dimension r over $\mathbb{Q}_p$, and A has entries in $\mathbb{Q}_p$, it follows that $A\otimes 1_K$ is isomorphic to the direct sum of r copies of A acting on $\mathbb{Q}_p^d$. Thus $h(A\otimes 1_K; K^d) = rh(A;\mathbb{Q}_p^d)$.

Since K contains the eigenvalues of $A\otimes 1_K$, we can put $A\otimes 1_K$ into its Jordan form

$$A\otimes 1_K \cong \bigoplus_{i=1}^{k} J(\lambda_i, d_i),$$

where $J(\lambda_i, d_i)$ denotes the Jordan block of size d_i corresponding to λ_i. To compute the entropy of Jordan blocks, we require the following lemma. Define as before $\log^+ x = \max\{\log x, 0\}$.

LEMMA 5.1. *With the above notations,* $h(J(\lambda,m);K^m) = rm\log^+|\lambda|_p$.

Proof. We first treat the case $m=1$. Let J denote $J(\lambda,1)=[\lambda]$. If μ is Haar measure on K, recall that $\operatorname{mod}_K(\lambda)$ is the number defined by $\mu(\lambda E) = \operatorname{mod}_K(\lambda)\mu(E)$ for measurable $E\subset K$ [We, p. 3]. Now $\operatorname{mod}_K(\lambda) = \operatorname{mod}_{\mathbb{Q}_p}(N_{K/\mathbb{Q}_p}(\lambda))$ [We, p. 7]. For $x\in\mathbb{Q}_p$ we have $\operatorname{mod}_{\mathbb{Q}_p}(x) = |x|_p$, and also $|\lambda|_p = |N_{K/\mathbb{Q}_p}(\lambda)|_p^{1/[K:\mathbb{Q}_p]}$ [K, p. 61]. If C is a compact ball centered at 0, then

$$\bigcap_{j=0}^{n-1} J^{-j}C = \begin{cases} J^{-(n-1)}C & \text{if } |\lambda|_p>1,\\ C & \text{if } |\lambda|_p\le 1.\end{cases}$$

It follows that

$$\lim_{n\to\infty} -\frac{1}{n}\log \mu\left(\bigcap_{j=0}^{n-1} J^{-j}C\right) = k(\mu, J)$$

exists, and equals $\log^+ \mathrm{mod}_K(\lambda) = r\log^+|\lambda|_p$. Since this also equals $h(J; K)$ by [**B**, Prop. 7], the case $m = 1$ is completed.

Next consider $J = J(\lambda, m)$ acting on K^m equipped with the sup norm. We may assume without loss that $|\lambda|_p > 1$, since otherwise both sides are 0. Since J commutes with multiplication by a power of p, by [**B**, Prop. 7] it is enough to show that

$$\lim_{n\to\infty} -\frac{1}{n}\log \mu\left(\bigcap_{j=0}^{n-1} J^{-j}C\right) = rm\log^+|\lambda|_p,$$

where C is the unit ball in K^m. Use of the non-archimedean nature of $|\cdot|_p$ makes the following p-adic computations about Jordan blocks simpler than the corresponding euclidean ones (see [**Wa**, Thm. 8.14] for the latter). Expansion of J^j shows that $J^jC \subset \lambda^{n-1}C$ for $0 \le j \le n-1$, so $\lambda^{-(n-1)}C \subset \bigcap_{j=0}^{n-1} J^{-j}C$. Expansion of J^{-j} shows that $J^{-(n-1)}C \subset \lambda^{-(n-1)}C$. We conclude that $\bigcap_{j=0}^{n-1} J^{-j}C = \lambda^{-(n-1)}C$. This proves the existence of the required limit, while its value is computed exactly as in the case $m = 1$. □

To complete the proof of Theorem 2, we note

$$h(A:\mathbb{Q}_p^d) = \frac{1}{r}h(A\otimes 1_K; K^d) = \frac{1}{r}\sum_{i=1}^{k} h(J(\lambda_i, d_i), K^{d_i})$$

$$= \frac{1}{r}\sum_{i=1}^{k} rd_i\log^+|\lambda_i|_p = \sum_{|\lambda_j|_p>1}\log^+|\lambda_j|_p. \qquad \square$$

6. *Derivation of Yuzvinskii's formula*

We use Theorems 1 and 2 to deduce Yuzvinskii's formula (2). Let $A \in GL(d, \mathbb{Q})$ have complex eigenvalues $\lambda_1, \dots, \lambda_d$, and let s be the least common multiple of the denominators of the coefficients of $\chi_A(t)$.

THEOREM 3 (Yuzvinskii). $h(A; \Sigma^d) = \log s + \sum_{|\lambda_j|>1}\log|\lambda_j|$.

Proof. Let $\chi_A(t) = t^d + a_1t^{d-1} + \cdots + a_d$. If p^e is the highest power of p dividing s, then

$$p^e = \max\{|a_1|_p, \dots, |a_d|_p, 1\}.$$

We will prove that $\log p^e = h(A; \mathbb{Q}_p^d)$. Then Theorem 1 shows that $\log s$ is just the sum over $p < \infty$ of the p-adic contributions to entropy, and this will complete the proof.

Factor $\chi_A(t) = \prod_{j=1}^d (t - \lambda_j^{(p)})$ over a finite extension of $\mathbb{Q}_p$, and order the eigenvalues so

$$|\lambda_1^{(p)}|_p \ge |\lambda_2^{(p)}|_p \ge \cdots \ge |\lambda_m^{(p)}|_p > 1 \ge |\lambda_{m+1}^{(p)}|_p \ge \cdots \ge |\lambda_d^{(p)}|_p.$$

If $|\lambda_j^{(p)}|_p \le 1$ for all j, then clearly $e = 0$ and $h(A; \mathbb{Q}_p^d) = 0$ as well. Thus we may suppose $|\lambda_1^{(p)}|_p > 1$. By the non-archimedean property of $|\cdot|_p$, we have by the specified ordering that

$$|a_m|_p = \left| \sum_{i_1<\cdots<i_m} \lambda_{i_1}^{(p)} \cdots \lambda_{i_m}^{(p)} \right|_p$$

$$= |\lambda_1^{(p)} \cdots \lambda_m^{(p)} + \text{smaller terms}|_p$$

$$= |\lambda_1^{(p)} \cdots \lambda_m^{(p)}|_p,$$

and by a similar calculation that all $|a_j|_p \le |a_m|_p$. Thus

$$p^e = \max_{1\le j\le d} \{|a_j|_p = \prod_{|\lambda_j^{(p)}|_p>1} |\lambda_j^{(p)}|_p.$$

By Theorem 2,

$$\log p^e = \sum_{|\lambda_j^{(p)}|_p>1} \log |\lambda_j^{(p)}|_p = h(A; \mathbf{Q}_p^d),$$

completing the proof. □

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Appendix B: The Entropy Formula

In this appendix we reproduce the proof of the entropy formula for the module $\mathbf{M} = \mathbf{R}/\mathbf{q}$. The proof is due to Doug Lind and Klaus Schmidt, and appears as Theorem 3.1 in Lind, Schmidt and Ward [1]. We also state here the addition formula for the entropy of a skew product of actions, also due to Doug Lind and Klaus Schmidt. A proof of this may be found in Appendix B of Lind, Schmidt and Ward [1].

Theorem B-1 (THE ADDITION FORMULA): If β is a measure-preserving $\mathbb{Z}^d$ action on the Lebesgue space (Ω, υ), α is a $\mathbb{Z}^d$ action by automorphisms on a compact group X with normalised Haar measure μ, and $\sigma : \mathbb{Z}^d \times \Omega \longrightarrow X$ is a cocycle for β and α, then:

$$h_{\upsilon\times\mu}(\beta \times_\sigma \alpha) = h_\upsilon(\beta) + h_\mu(\alpha).$$

By a cocycle for β and α, we mean a map σ satisfying the cocycle relation

$$\sigma(\mathbf{a} + \mathbf{b}, \omega) = \alpha_{\mathbf{a}}(\sigma(\mathbf{b}, \omega)) + \sigma(\mathbf{a}, \beta_{\mathbf{b}}\omega)$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$ and $\omega \in \Omega$, where $+$ is the group operation in X. This cocycle equation means that there is a well-defined skew product action $\beta \times_\sigma \alpha$ on the product space $(\Omega \times X, \upsilon \times \mu)$ given by $(\beta \times_\sigma \alpha)_{\mathbf{n}}(\omega, x) = (\beta_{\mathbf{n}}\omega, \alpha_{\mathbf{n}}x + \sigma(\mathbf{n}, \omega))$.

Write $\mathbf{R}$ for the ring of Laurent polynomials in d variables with integer coefficients, for some d fixed throughout the proof. Write $\mathbf{R}(k)$ for the ring in k variables for any k other than d. The *girth* of the rectangle

$$Q = [b_1, \ldots, b_1 + c_1 - 1] \times \ldots \times [b_d, \ldots, b_d + c_d - 1]$$

is defined to be $g(Q) = \min\{c_1, \ldots, c_d\}$.

Theorem B-2 (THE ENTROPY FORMULA): The joint topological entropy of the action $\alpha_{\mathbf{R}/\mathbf{q}}$ on the group $X_{\mathbf{R}/\mathbf{q}}$ is given by

$$h(\alpha_{R/q}) = \begin{cases} 0 \text{ if } \mathbf{q} \text{ is non-principal,} \\ \infty \text{ if } \mathbf{q} = \{0\}, \text{ and} \\ \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i s_1}, \ldots, e^{2\pi i s_d})| \, ds_1 \ldots ds_d \text{ if } \mathbf{q} = \langle f \rangle,\ f \neq 0. \end{cases}$$

Proof. For the case $d = 1$, this is equivalent to Yuzvinskii's formula as pointed out in Theorem II.1.9.

If the ideal $\mathbf{q}$ is trivial, then the action is the full d dimensional shift with alphabet given by the circle group, so the entropy is infinite.

If the ideal $\mathbf{q}$ is non-principal, then we may find polynomials f and g in $\mathbf{q}\setminus\{0\}$ with the property that f and g have no factor in common. Let $\mathbf{p} = \langle f \rangle$, $\mathbf{p}' = \langle g \rangle$ and $\mathbf{r} = \mathbf{p} + \mathbf{p}' = \langle f, g \rangle$. Let $\varphi : \mathbf{R}/\mathbf{p} \longrightarrow \mathbf{R}/\mathbf{p}$ denote multiplication by g. Since g and f have no factor in common, φ is a monomorphism. The image of φ is $g(\mathbf{R}/\mathbf{p}) = \mathbf{p}'/\mathbf{p}\mathbf{p}' = \mathbf{p}'/(\mathbf{p} \cap \mathbf{p}') \cong \mathbf{p}+\mathbf{p}'/\mathbf{p} = \mathbf{r}/\mathbf{p}$. We therefore have an exact sequence of $\mathbf{R}$-modules,

$$0 \longrightarrow \mathbf{R}/\mathbf{p} \xrightarrow{\varphi} \mathbf{R}/\mathbf{p} \longrightarrow \mathbf{R}/\mathbf{r} \longrightarrow 0$$

and so, by the entropy addition formula, $h(\alpha_{\mathbf{R}/\mathbf{r}}) = 0$. Now $\mathbf{r} \subset \mathbf{q}$ and the epimorphism $\mathbf{R}/\mathbf{r} \longrightarrow \mathbf{R}/\mathbf{q}$ is dual to an inclusion $X_{\mathbf{R}/\mathbf{q}} \longrightarrow X_{\mathbf{R}/\mathbf{r}}$ so $h(\alpha_{\mathbf{R}/\mathbf{q}}) = 0$.

We are therefore left with the case $\mathbf{q} = \langle f \rangle$ for some $f \neq 0$, and $d \geq 2$. For brevity write $\log M(f)$ for the expression given in the statement of the theorem: $M(f)$ is then the *Mahler measure* of the polynomial f.

<u>First inequality</u>: $h(\alpha_{\mathbf{R}/\mathbf{q}}) \geq \log M(f)$.

For $s, t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, let $|s - t| = |s + \mathbb{Z}, t + \mathbb{Z}|_{\mathbb{R}}$. For $x, y \in \mathbb{T}^{\mathbb{Z}^d}$ let

$$\rho(x, y) = \sum_{n \in \mathbb{Z}^d} |x(\mathbf{n}) - y(\mathbf{n})| \times 2^{-|\mathbf{n}|}$$

where $|\mathbf{n}| = \max\{|n_1|, \ldots, |n_d|\}$.

Lemma 1: For any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ and an integer $b = b(\varepsilon) > 0$ such that $\rho(x, y) > \delta$ implies that $|x(\mathbf{j}) - y(\mathbf{j})| > \varepsilon$ for some $\mathbf{j}$ with $|\mathbf{j}| < b$.

Proof. Let $\sum_{\mathbf{j} \in \mathbb{Z}^d} 2^{-|\mathbf{j}|} = K < \infty$. Choose $b(\varepsilon)$ large enough to ensure that $\sum_{|\mathbf{j}|>b} 2^{-|\mathbf{j}|} < \varepsilon$, and put $\delta(\varepsilon) = (K+1)\varepsilon$. If $|x(\mathbf{j}) - y(\mathbf{j})| \le \varepsilon$ for all $|\mathbf{j}| < b(\varepsilon)$, then

$$\rho(x, y) \le \varepsilon + \sum_{|\mathbf{j}|<b} 2^{-|\mathbf{j}|}\varepsilon = (K+1)\varepsilon = \delta(\varepsilon). \ \square$$

Thus, for any $\varepsilon > 0$, there is a $b > 0$ and a $\delta > 0$ with the property that for every rectangle Q in $\mathbb{Z}^d$ with sides of length l_j, and for every (Q, δ) separated set $F \subset \mathbb{T}^{\mathbb{Z}^d}$, there is a smaller rectangle, Q', with sides of length $l_j - 2b$ such that F is (Q', ε) separated uniformly (in the maximum distance over the co-ordinates of Q'). This means that the entropy $h(\alpha)$ may be computed using sets that are separated in the maximum metric.

Let f be the polynomial $f(\mathbf{u}) = \sum c_\mathbf{j} \mathbf{u}^\mathbf{j}$. For an integer vector $\mathbf{r} \in \mathbb{Z}^d$, we can define a polynomial of one variable by setting

$$f_\mathbf{r}(u) = f(u^{r_1}, \ldots, u^{r_d}) = \sum_{\mathbf{j} \in \mathbb{Z}^d} c_\mathbf{j} u^{\mathbf{r}\cdot\mathbf{j}}.$$

Let $q(\mathbf{r}) = \min\{|\mathbf{m}| \mid \mathbf{m} \neq 0 \text{ and } \mathbf{m}\cdot\mathbf{r} = 0\}$. We quote the following result.

Proposition 2: (Lawton [2]) For every polynomial $f \in \mathbf{R}$,

$$\lim_{q(\mathbf{r}) \to \infty} M(f_\mathbf{r}) = M(f).$$

An integer vector $\mathbf{r} \in \mathbb{Z}^d$ is *primitive* if the greatest common divisor of its components is one. For a primitive $\mathbf{r} \in \mathbb{Z}^d$, define a map $\psi_\mathbf{r}\colon \mathbb{T}^{\mathbb{Z}} \longrightarrow \mathbb{T}^{\mathbb{Z}^d}$ by $(\psi_\mathbf{r}\mathbf{x})(\mathbf{j}) = \mathbf{x}(\mathbf{r}\cdot\mathbf{j})$ for $\mathbf{x} \in \mathbb{T}^{\mathbb{Z}}$. If $\psi_\mathbf{r}\mathbf{x} = \psi_\mathbf{r}\mathbf{y}$ then $\mathbf{x}(\mathbf{r}\cdot\mathbf{j}) = \mathbf{y}(\mathbf{r}\cdot\mathbf{j})$ for all $\mathbf{j} \in \mathbb{Z}^d$, so $(\mathbf{x} - \mathbf{y})(n)$ must vanish for all $n \in d\cdot\mathbb{Z}$ where $d = \text{g.c.d.}(r_1, \ldots, r_d)$. This means that $\psi_\mathbf{r}$ is injective if $\mathbf{r}$ is primitive.

For brevity, write (X, α) for $(X_{R/q}, \alpha_{R/q})$, and $(X_\mathbf{r}, \alpha_\mathbf{r})$ for the one dimensional system $(X_{R(1)/<f_\mathbf{r}>}, \alpha_{R(1)/<f_\mathbf{r}>})$. We claim that $\psi_\mathbf{r}(X_\mathbf{r}) \subset X$. This may be seen as follows: if $\mathbf{x} \in X_\mathbf{r}$ then $\sum_{\mathbf{j}\in\mathbb{Z}^d} c_\mathbf{j}(\psi_\mathbf{r}\mathbf{x})(\mathbf{n} + \mathbf{j}) = \sum_{\mathbf{j}\in\mathbb{Z}^d} c_\mathbf{j}\mathbf{x}(\mathbf{r}\cdot\mathbf{n} + \mathbf{r}\cdot\mathbf{j}) = 0$ by the definition of $f_\mathbf{r}$.

For $n \geq 1$, let $\mathbf{r}_n = (1, n, n^2, \ldots, n^{d-1})$. This is primitive for all $n \geq 1$, and $q(\mathbf{r}_n) = n \longrightarrow \infty$ in n. Let $Q_{n,m} = \{0, 1, \ldots, n-1\}^{d-1} \times \{0, 1, \ldots, m-1\}$. The map $\mathbf{j} \longmapsto \mathbf{r}_n\cdot\mathbf{j}$ is a bijection from $Q_{n,m}$ to $\{0, 1, \ldots, |Q_{n,m}| - 1\}$. Let $b_1(\varepsilon)$ be the integer determined by lemma 1, and set $P_{n,m} = \{b_1(\varepsilon), \ldots, |Q_{n,m}| - b_1(\varepsilon) - 1\}$. Fix $\theta > 0$. Let $\delta_1(\varepsilon)$ be determined by lemma 1, so that $\delta_1(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. From the definition of topological entropy, there exists an $\varepsilon > 0$ with the property that for all sufficiently large n, there is an $m_0(\varepsilon, \theta, n) > n$ for which $m > m_0$ implies that there exists a $(P_{n,m}, \delta_1(\varepsilon))$ - separated set $F \subset X_{\mathbf{r}_n}$ with

$$|F| > \exp[|P_{n,m}| \times (h(\alpha_{\mathbf{r}_n}) - \theta)].$$

Suppose that $\mathbf{x}$ and $\mathbf{y}$ are distinct points in F, and let $x = \psi_{\mathbf{r}_n}(\mathbf{x})$, $y = \psi_{\mathbf{r}_n}(\mathbf{y})$. By separation, there exists an $i \in P_{n,m}$ with $\rho(\alpha_{\mathbf{r}_n}{}^i\mathbf{x}, \alpha_{\mathbf{r}_n}{}^i\mathbf{y}) > \delta_1(\varepsilon)$. By lemma 1, separation implies that there is a j with $0 \leq j < |Q_{n,m}|$ for which $|\mathbf{x}(j) - \mathbf{y}(j)| > \varepsilon$. There is a unique $\mathbf{k} \in Q_{n,m}$ with $\mathbf{r}_n\cdot\mathbf{k} = j$, so $|x(\mathbf{k}) - y(\mathbf{k})| > \varepsilon$, and $\rho(\alpha_\mathbf{k}x, \alpha_\mathbf{k}y) > \varepsilon$. This means that $\psi_{\mathbf{r}_n}(F) \subset X$ is a $(Q_{n,m}, \varepsilon)$ - separated set for α with the same cardinality as F. Thus, for any $m \geq m_0(\varepsilon, n, \theta)$,

$$\frac{1}{|Q_{n,m}|} \log s_{Q_{n,m}}(\varepsilon, \alpha) \geq \frac{|P_{n,m}|}{|Q_{n,m}|} (h(\alpha_{\mathbf{r}_n}) - \theta).$$

Now the girth $g(Q_{n,m}) \longrightarrow \infty$ as $n \longrightarrow \infty$, and $|P_{n,m}|/|Q_{n,m}| \longrightarrow 1$ as $n \longrightarrow \infty$. From proposition 1 and the $d = 1$ case we see that, on taking the limsup in n, $s(\varepsilon, \alpha) \geq \log M(f) - \theta$ for every $\theta > 0$. Hence $h(\alpha) \geq \log M(f)$, completing the proof of the first inequality.

Second inequality: $h(\alpha_{R/q}) \le \log M(f)$.

In order to prove the upper bound, we first transform f into a more convenient form. If $f(\mathbf{u}) = \sum c_j \mathbf{u}^j$ and $A \in GL(d, \mathbb{Z})$, let $f^A(\mathbf{u}) = \sum c_j \mathbf{u}^{Aj}$. Then $\alpha_{R/\langle f^A \rangle}$ is conjugate to $\alpha^A{}_{R/\langle f \rangle}$, where the action α^A is defined by $(\alpha^A)_n = \alpha_{An}$. It is clear that $h(\alpha^A) = h(\alpha)$, and, since A preserves Haar measure on $\mathbb{T}^d$, $M(f^A) = M(f)$. Monomials are units in the ring $\mathbf{R}$ so multiplying by a monomial is a module automorphism and does not affect the action or the entropy.

By applying a suitable $A \in GL(d, \mathbb{Z})$ and multiplying by a monomial, we can arrange that f has the form:

$$f(u_1, \ldots, u_d) = q \cdot u_d^D + f_{D-1}(u_1, \ldots, u_{d-1}) u_d^{D-1} + \ldots + f_0(u_1, \ldots, u_{D-1}), \quad (*)$$

where $q \in \mathbb{Z} \backslash \{0\}$, each $f_k \in \mathbb{Z}[u_1, \ldots, u_{d-1}]$, and $f_0 \neq 0$. If f is in this form and $f = \sum c_j \mathbf{u}^j$, then there is a $p > 0$ such that

$$\{j \in \mathbb{Z}^d \mid c_j \neq 0\} \subset (\{0, 1, \ldots, p-1\}^{d-1} \times \{0, 1, \ldots, D-1\}) \cup \{De_d\}.$$

Lemma 3: Suppose that $f \in \mathbf{R}$ has the form $(*)$ above, and let $S = \mathbb{Z}^{d-1} \times \{0, 1, \ldots, D-1\}$. Then the homomorphism $X_{R/\langle f \rangle} \longrightarrow \mathbb{T}^S$ given by coordinate restriction is surjective.

Proof. If $x(j) \in \mathbb{T}$ is chosen arbitrarily for all $j \in S$, we need to check that x can be extended to a point of $X_{R/\langle f \rangle}$. If $\mathbf{u}' = (u_1, \ldots, u_{d-1})$, write $f_k(\mathbf{u}') = \sum_{\mathbf{i} \in \mathbb{Z}^{d-1}} c_k(\mathbf{i})(\mathbf{u}')^{\mathbf{i}}$. By $(*)$, any extension of x to $\mathbb{Z}^{d-1} \times \{De_d\}$ must satisfy, for $\mathbf{i} \in \mathbb{Z}^{d-1}$, the equation

$$qx(\mathbf{i}, D) = -\sum_{k=0}^{D-1} \sum_{\mathbf{m} \in \mathbb{Z}^{d-1}} c_k(\mathbf{m}) x(\mathbf{i} + \mathbf{m}, k).$$

This shows that such an extension is always possible, and for every $\mathbf{i} \in \mathbb{Z}^{d-1}$ there are q choices for the additional coordinate $x(\mathbf{i}, D)$. Repeating this procedure extends x to $\mathbb{Z}^{d-1} \times \{0, 1, \ldots\}$. To extend in the other direction, to $\mathbb{Z}^{d-1} \times \{-1\}$, we must find values $x(\mathbf{i}, -1)$ for $\mathbf{i} \in \mathbb{Z}^{d-1}$, so that

$$\sum_{\mathbf{m}\in\mathbb{Z}^{d-1}} c_0(\mathbf{m})x(\mathbf{i}+\mathbf{m}, -1) = -\sum_{k=1}^{D-1} \sum_{\mathbf{m}\in\mathbb{Z}^{d-1}} c_k(\mathbf{m})x(\mathbf{i}+\mathbf{m}, k-1) - qx(\mathbf{i}, D-1).$$

Since $\mathbf{R}(d-1)$ is an integral domain and $f_0 \neq 0$, the homomorphism $\mathbf{R}(d-1)^* \longrightarrow \mathbf{R}(d-1)^*$ dual to multiplication by f_0 is surjective, so the extension exists. Repeated application of this argument fills out the value of x to all co-ordinates, producing a point in $X_{\mathbf{R}/\langle f\rangle}$. □

Let Y_n denote the set of points in $X_{\mathbf{R}/\langle f\rangle}$ which have period n in each of the first $(d-1)$ coordinates. Then Y_n is a closed α - invariant subgroup. The following remarks will show that there is a constant $a > 0$ independent of n such that, for every $m > 0$, the projection of Y_n onto the coordinates

$$\{0, 1, \ldots, n - am - 1\}^{d-1} \times \{0, 1, \ldots, m-1\}$$

coincides with the projection of $X_{\mathbf{R}/\langle f\rangle}$ onto these coordinates. This allows our replacement of $X_{\mathbf{R}/\langle f\rangle}$ by Y_n, and $\alpha_{\mathbf{R}/\langle f\rangle}$ by an automorphism A of the finite dimensional torus Y_n induced by the shift in the last (d^{th}) coordinate.

Let n be a positive integer (large compared to p). Put $Q_{/n} = (\mathbb{Z}/n\mathbb{Z})^{d-1}$, $Q_{/n,D} = Q_{/n} \times \{0, \ldots, D-1\}$, and $\mathbb{T}_n = \mathbb{T}^{Q_{/n}}$, $\mathbb{T}_{n,D} = \mathbb{T}^{Q_{/n,D}}$. If we identify $Q_{/n,D}$ with $\{0, \ldots, n-1\}^{d-1} \times \{0, \ldots, D-1\} \subset \mathbb{Z}^d$, then lemma 3 shows that the homomorphism $\varphi: X_{\mathbf{R}/\langle f\rangle} \longrightarrow \mathbb{T}_{n,D}$ given by $\varphi(x) = x|_{Q_{/n,D}}$ is surjective. Thus if $\upsilon = \upsilon_{n,D}$ is normalized Haar measure on $\mathbb{T}_{n,D}$, and μ is normalized Haar measure on $X_{\mathbf{R}/\langle f\rangle}$ then $\varphi^*(\mu) = \upsilon$.

We shall assume from now on that $f \in \mathbf{R}$ has the form (*), and that p is chosen as above. To simplify notation, we shall also assume that $q = 1$ in the expression (*).

There is a homomorphism $A: \mathbb{T}_{n,D} \longrightarrow \mathbb{T}_{n,D}$ given by

$$(Ay)(\mathbf{i}, k) = \begin{cases} y(\mathbf{i}, k+1) & \text{if } 0 \le k \le D-2, \\ -\sum_{j=0}^{D-1} \sum_{\mathbf{m} \in \{0,\ldots,p-1\}^{d-1}} c_j(\mathbf{m}) y(\mathbf{i} + \mathbf{m}, j) & \text{if } k = D-1. \end{cases}$$

The relationship between A and α is that the orbit of a point $y \in \mathbb{T}_{n,D}$ under A and the extension of y to $X_{R/\langle f \rangle}$ given by lemma 3 must agree on the interior of a rectangle in the following sense. Let

$$P_{n,m} = \{0, \ldots, n-1-pm\}^{d-1} \times \{0, \ldots, m-1\},$$

and suppose $(\mathbf{i}, k) \in P_{n,m}$. If $y \in \mathbb{T}_{n,D}$ and x is any point in $X_{R/\langle f \rangle}$ with $\varphi(x) = y$, then $x(\mathbf{i}, k) = (A^k y)(\mathbf{i} \bmod n, 0)$. This observation is the basis of the proof for the upper bound.

Fix an $\varepsilon_1 > 0$. Using the proof of lemma 1, let $\varepsilon = \varepsilon_1/(2^d + 1)$, and $b = b(\varepsilon)$. If

$$R_{n,m} = \{b, \ldots, n-1-pm-b\}^{d-1} \times \{b, \ldots, m-1-b\},$$

then

$$\{x \in X_{R/\langle f \rangle} \mid |x(\mathbf{j})| < \varepsilon, \mathbf{j} \in P_{n,m}\} \subset \bigcap_{\mathbf{j} \in R_{n,m}} \alpha_{-\mathbf{j}} B(\varepsilon_1) = D_{R_{n,m}}(\alpha, \varepsilon_1).$$

For $y \in \mathbb{T}_{n,D}$, put $\|y\|_\infty = \max \{|y(\mathbf{j})| \mid \mathbf{j} \in Q_{/n,D}\}$. By the observation in the previous paragraph, we have

$$\varphi^{-1}\{y \in \mathbb{T}_{n,D} \mid \|A^j y\|_\infty < \varepsilon \text{ for } 0 \le j < m\} \subset \{x \in X_{R/\langle f \rangle} \mid |x(\mathbf{j})| < \varepsilon, \mathbf{j} \in P_{n,m}\}.$$

Thus, in order to obtain an upper bound for $(-1/|R_{n,m}|) \log \mu(D_{R_{n,m}}(\alpha, \varepsilon_1))$, it suffices to obtain one for

$$\frac{-1}{|R_{n,m}|} \log \upsilon\{y \in \mathbb{T}_{n,D} \mid \|A^j\|_\infty < \varepsilon, 0 \le j < m\}. \qquad (+)$$

Fix a sequence $m(n) \longrightarrow \infty$ in such a way that $m(n)/\log(n) \longrightarrow \infty$ and $m(n)/n \longrightarrow 0$ as $n \longrightarrow \infty$. Then $g(R_{n,m}) \longrightarrow \infty$, and $|R_{n,m}|/|Q_{n,m}| \longrightarrow 1$, so in (+) we may replace $R_{n,m}$ with $Q_{/n,m(n)}$.

We estimate the measure in (+) as follows. If $\mathbb{C}_n = \mathbb{C}^{Q/n}$, then A induces a linear map on $\mathbb{C}_n{}^D$. We will decompose $\mathbb{C}_n{}^D$ into an orthogonal direct sum of D-dimensional A - invariant subspaces indexed by $Q_{/n}$. By adding up the volume decrease from the intersection in (+) over these subspaces we obtain a Riemann sum approximation to $\log M(f)$, from which the upper bound will follow.

Use the symbol A for the linear map on $\mathbb{C}_n{}^D$ induced by the homomorphism A of $\mathbb{T}_{n,D}$. Let $\|\cdot\|_\infty$ be the sup norm on $\mathbb{C}_n{}^D$, and let $\upsilon_{\mathbb{C}}$ be Haar measure on $\mathbb{C}_n{}^D$ normalized so the unit cube has measure 1. Then we have (**):

$$\upsilon\{y \in \mathbb{T}_{n,D} \mid \|A^j y\|_\infty < \varepsilon \text{ for } 0 \le j < m\} = (\upsilon_{\mathbb{C}}\{z \in \mathbb{C}_n^D \mid \|A^j z\|_\infty < \varepsilon \text{ for } 0 \le j < m\})^{1/2}.$$

The square root is taken to compensate for the doubling in real dimension due to passing to complex vector spaces.

For $1 \le j \le d$ let P_j act on $\mathbb{C}_n$ by $(P_j z)(\mathbf{i}) = z(\mathbf{i} + \mathbf{e}_j \bmod n)$, and put $\mathbf{P} = (P_1, \ldots, P_{d-1})$. Then the matrix of A with respect to the standard basis of $\mathbb{C}_n{}^D$ is

$$A = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & I \\ -f_0(\mathbf{P}) & \cdot & \cdot & \cdots & -f_{D-1}(\mathbf{P}) \end{bmatrix}.$$

Fix a primitive n^{th} root of unity, $\omega = e^{2\pi i/n}$. For $\mathbf{k} \in Q_{/n}$ define a vector

$\upsilon_k \in \mathbb{C}_n$ by setting $\upsilon_k(j) = \omega^{j.k}/\sqrt{|Q/_n|}$ for each $j \in Q/_n$. The set $\{\upsilon_k \mid k \in Q/_n\}$ define an orthonormal basis for $\mathbb{C}_n$. Also, $P_j(\upsilon_k) = \omega^{kj} \times \upsilon_k$ so the vectors υ_k are simultaneous eigenvectors for the P_j.

Let $W_k = (\mathbb{C}\upsilon_k)^D$, equipped with the usual Euclidean metric. Then $\mathbb{C}_n{}^D = \bigoplus_{k \in Q/_n} W_k$ is an orthogonal direct sum. Furthermore, each W_k is an A-invariant D dimensional complex subspace, and the matrix of A with respect to the standard basis on W_k is

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ . & . & . & & 1 \\ -f_0(\eta) & . & . & \dots & -f_{D-1}(\eta) \end{bmatrix}.$$

where $\eta = (\omega^{k_1}, \dots, \omega^{k_{d-1}})$ for $k \in Q/_n$.

For each $k \in Q/_n$ let μ_k be Haar measure on W_k normalised so the unit cube has measure 1. Since the decomposition of $\mathbb{C}_n{}^D$ is orthogonal, $\upsilon_{\mathbb{C}} = \prod_k \mu_k$. Denote the unit ball in W_k by B_k, and put $B_* = \oplus_k B_k$. Let $B_\infty(\varepsilon)$ be the ε-ball in $\mathbb{C}_n{}^D$ with respect to the $\|\cdot\|_\infty$ norm. Since each coordinate of υ_k has modulus $n^{-(d-1)/2}$, and there are $n^{(d-1)}$ such vectors, we have that $B_\infty(\varepsilon) \supset \varepsilon n^{-(d-1)} B_*$. Hence

$$\{z \in \mathbb{C}_n{}^D \mid \|A^j z\|_\infty < \varepsilon,\ 0 \le j < m(n)\}$$

$$= \bigcap_{j=0,\dots,m(n)-1} A^{-j} B_\infty(\varepsilon) \supset \varepsilon n^{-(d-1)} \bigoplus_{k \in Q/_n} \left(\bigcap_{j=0,\dots,m(n)-1} A_k{}^{-j} B_k\right),$$

and so by (**),

$$\frac{-1}{|Q_{n,m(n)}|} \log \upsilon\{y \in \mathbb{T}_{n,D} \mid \|A^j y\|_\infty < \varepsilon, 0 \le j < m(n)\}$$

$$\le \frac{1}{|Q_{n,m(n)}|} \times \frac{1}{2} \times \log \upsilon_{\mathbb{C}}(\varepsilon n^{-(d-1)} \oplus_{k \in Q_{/n}} (\textstyle\bigcap_{j=0,\dots,m(n)-1} A_k^{-j} B_k))$$

$$= \frac{\log \varepsilon^{-1} + (d-1)D|Q_{/n}| \log n}{2m(n) \times |Q_{/n}|}$$

$$+ \frac{1}{2|Q_{/n}|} \sum_{k \in Q_{/n}} \frac{-1}{m(n)} \log \mu_k(\textstyle\bigcap_{j=0,\dots,m(n)-1} A_k^{-j} B_k)$$

$$= O\left(\frac{\log n}{m(n)}\right) + \frac{1}{2|Q_{/n}|} \sum_{k \in Q_{/n}} b_{m(n)}(A_k),$$

where the summands $b_{m(n)}$ of the last line are defined by the summands in the previous line.

The last line of the above expression contains a Riemann sum approximation. In order to check that this converges to the integral defining $\log M(f)$ we require a uniformity result about the entropy of linear maps on $\mathbb{C}^D$. For $T \in \mathbb{C}^{D\times D}$ let $h(T)$ denote the Bowen entropy of T, as defined for instance in Chapter 7 of Walters [1]. The unit cube in $\mathbb{C}^D$ is taken to be the usual fundamental domain for $(\mathbb{Z} + i\mathbb{Z})^D$.

Lemma 4: Let B be the unit ball in $\mathbb{C}^D$ for the Euclidean norm, and μ_D be Haar measure on $\mathbb{C}^D$ normalised so that the unit cube has measure 1. For $T \in \mathbb{C}^{D\times D}$ put

$$b_m(T) = (-1/m) \log \mu_D(\textstyle\bigcap_{j=0,\dots,m-1} T^{-j} B),$$

$$\text{and } h(T) = \sum_{k=1}^{D} \log^{+}|\lambda_k|^2 = 2 \times \int_0^1 \log|\chi_T(e^{2\pi it})|\,dt,$$

where T has characteristic polynomial χ_T and eigenvalues λ_k counted with multiplicity.Then:

(i) $h(T)$ and each $b_m(T)$ are continuous functions of T.

(ii) $b_m(T) \longrightarrow h(T)$ uniformly on compact subsets of $\mathbb{C}^{D\times D}$.

Proof. A detailed proof of this statement is given in Lind, Schmidt and Ward [1], Lemma 3.5. □

Write S for the unit circle in $\mathbb{C}$. For $s \in S^{d-1}$ let

$$A(s) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ . & . & . & & 1 \\ -f_0(s) & . & . & \dots & -f_{D-1}(s) \end{bmatrix}.$$

This is in companion form, and the characteristic equation is $f(s, u)$ (by (*)). Then $A_k = A(\omega^k)$, and the Riemann sum approximation approximates the integral

$$\frac{1}{2} \int_{\mathbb{T}^{d-1}} h(A(e^{2\pi it}))\,dt = \int_{\mathbb{T}^{d-1}} \int_0^1 \log|f(e^{2\pi it}, e^{2\pi is})|\,ds\,dt = \log M(f). \quad (++)$$

To complete the proof of the theorem, let $\delta > 0$. Since $\{A(e^{2\pi it}) \mid t \in \mathbb{T}^{d-1}\}$ is a compact subset of $\mathbb{C}^{D\times D}$, lemma 4 and (++) together imply that, for sufficiently large n,

$$\frac{1}{2 \times |Q_{/n}|} \sum_{k \in Q_{/n}} b_{m(n)}(A_k) = \frac{1}{2 \times |Q_{/n}|} \sum_{k \in Q_{/n}} b_{m(n)}(A(e^{2\pi i k/n}))$$

$$< \frac{1}{2} \int_{\mathbb{T}^{d-1}} h(A(e^{2\pi i t}))\, dt + \delta = \log M(f) + \delta. \quad (***)$$

From the definition of topological entropy and (**), it follows that

$$s_{R_{n,m(n)}}(\varepsilon_1, \alpha) \le \frac{|Q_{n,m(n)}|}{|R_{n,m(n)}|} \left(O\left(\frac{\log n}{m(n)}\right)\right) + \log M(f) + \delta).$$

Since $m(n)/n \longrightarrow 0$ we have that $|Q_{n,m(n)}|/|R_{n,m(n)}| \longrightarrow 1$, and since $\log n/m(n) \longrightarrow 0$, for large enough n the right hand side of the above expression is dominated by $\log M(f) + 2\delta$. Now $g(R_{n,m(n)}) \longrightarrow \infty$ as $n \longrightarrow \infty$, so the convergence in the definition of topological entropy (details in Proposition A-2 of Lind, Schmidt and Ward [1]) implies that $s(4\varepsilon_1, \alpha) \le \log M(f) + 2\delta$. By letting $\varepsilon_1 \longrightarrow 0$ and then $\delta \longrightarrow 0$ we conclude that $h(\alpha) \le \log M(f)$, which is the upper bound under the assumption that $|q| = 1$.

If $|q| > 1$, then the estimate (***) for the linear map $q^{-1}A$ gives an upper bound of $\log M(q^{-1}f) + \delta$. The $|q|$ possible choices in the forward extension of lemma 3 contribute an additional term $\log |q|$ to the volume decrease. Since $\log M(q^{-1}f) = \log M(f) - \log |q|$, we again obtain the upper bound $h(\alpha) \le \log M(f)$. □

Appendix C: Calculation of Periodic Points

In this appendix we do some explicit calculations to find the number of points of given period in some simple systems in *Id(2).* For elements of *Id(1)* - toral or solenoidal automorphisms - it is easy to write down the number of points of given period (cf. Lemma II.3.2).

Remark C-1: Given $X = X_f$, the points of period $\mathbf{n}$ may be calculated by using unit roots as in lemma II.3.3.

There is of course a direct method for the calculation of $|\mathrm{Fix}_f(\mathbf{n})|$:- the group J_n is simply a quotient group of the additive group $\mathbf{R}$ and the volume of a fundamental domain is the determinant of the matrix $M_n(f)$ whose rows are the co-ordinates in $\mathbb{Z}^d$ of the vectors generating the stabiliser group $H_x = \{ \mathbf{n} \in \mathbb{Z}^d \mid \alpha_n x = x \}$. That is, if we order the set of monomials:-

$$\mathcal{M}(\mathbf{n}) = \{ u_1^{m_1} \dots u_d^{m_d} \mid 0 \le m_i \le n_i \}$$

in any way, then we can write down the vectors with respect to this ordered basis of all the polynomials of the form $\mathbf{u}^{\mathbf{n}}f$, reduced according to the rules $u_i^{n_i} = 1$, and choose a spanning set $v_1, \dots, v_p$ to give the matrix $M_n(f) = [v_1| \dots |v_p]$. Notice that $p = n_1 \dots n_d$ is the cardinality of $\mathcal{M}(\mathbf{n})$ so this method is computationally much harder than the method of Remark C.1. Another problem with this method is the arbitrary choice of basis for $\mathcal{M}$. This prevents (or certainly masks) any kind of recursive behaviour in M_n (and therefore in $|\mathrm{Fix}(\mathbf{n})|$) as $\mathbf{n}$ is changed. For $d = 1$ there is a canonical ordering that extends well: $1 < x < x^2 < x^3 < \dots$. We illustrate both this and the methods with an example.

Example C-2: Consider the case $I = \langle f(x, y) \rangle$ where $f(x, y) = 3 + x + y + x^2$. We will calculate the values of $|\mathrm{Fix}_f(3, 3)|$, $|\mathrm{Fix}_f(1, 9)|$ and $|\mathrm{Fix}_f(9, 1)|$.

(i) Elimination method. Eliminate y from the pair $y^3 = 1$ and $f(x, y) = 0$ to arrive at the polynomial $f_3(x) = x^6 + 3x^5 + 12x^4 + 19x^3 + 36x^2 + 27x + 28$. Then evaluate the number of points of period three in the solenoid defined by this polynomial to conclude that:-

$$|\mathrm{Fix}_f(3,3)| = \prod_{j=0}^{2} \left|\left(\frac{-1}{2} + \frac{1}{2}\sqrt{-11-\omega^j}\right)^3 - 1\right| \times \prod_{j=0}^{2} \left|\left(\frac{-1}{2} - \frac{1}{2}\sqrt{-11-\omega^j}\right)^3 - 1\right|$$

$$= 10\,206$$

Where ω is a primitive cube root of unity.

(ii) Matrix method. Place the following ordering on $\mathcal{M}(3, 3)$:-

$$1 < x < y < x^2 < y^2 < xy < x^2y < xy^2 < x^2y^2$$

Then the matrix is given by:-

$$M_{(3,3)}(f) = \begin{bmatrix} 3 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 3 \end{bmatrix}$$

The determinant of this matrix is found to be 10 206.

Similarly, by the elimination method we find that:-

$$|\mathrm{Fix}_f(9,1)| = \left|\left(\frac{1}{2}\right)^9(-1 + \sqrt{-15})^9\right| \times \left|\left(\frac{1}{2}\right)^9(-1 - \sqrt{-15})^9\right| = 261\,364$$

$$|\mathrm{Fix}_f(1,9)| = 5^9 - (-1)^9 = 1\,953\,126$$

Notice that the asymmetry of the polynomial f is reflected in these quantities.

We give tables of numbers of periodic points of small periods for the four two-dimensional systems determined by the polynomials $x + y$, $1 + x + y$, $2 + x + y$ and $3 + x + y$. Since these polynomials are symmetric $|\mathrm{Fix}(n, m)| = |\mathrm{Fix}(m, n)|$ and only half of each table has been filled in. We also give the set of bad periods $L_{\langle f \rangle}$ for each example: this is the maximal lattice L in $\mathbb{Z}^2$ with the property that for any $(\omega_1, \omega_2) \in V_{\mathbb{C}}(f) \cap \mathcal{U}$, $\omega_1{}^n = \omega_2{}^m = 1$ for all $(n, m) \in L$. The tables were drawn up with the assistance of a computer program kindly written for me by Maria Iano.

Examples C-3: Consider the system $(X_{\langle f \rangle}, \mathbb{Z}^2)$.

(i) Let $f(x, y) = x + y$. Then $V(f)$ contains the unit roots $(1, -1)$ and $(-1, 1)$ whose orders are $(1, 2)$ and $(2, 1)$ respectively. We therefore expect that there will be infinitely many points of periods with even sides. The set of bad periods is $L_{\langle x+y \rangle} = (2\mathbb{Z} \times \mathbb{Z}) \cup (\mathbb{Z} \times 2\mathbb{Z})$. This example has zero entropy and this is reflected in the fact that it has very few periodic points for good periods.

	1	2	3	4	5	6	7	8
1	2	Infinite	2	Infinite	2	Infinite	2	Infinite
2		Infinite	Infinite	Infinite	Infinite	Infinite	Infinite	Infinite
3			8	Infinite	2	Infinite	2	Infinite
4				Infinite	Infinite	Infinite	Infinite	Infinite
5					32	Infinite	2	Infinite
6						Infinite	Infinite	Infinite
7							128	Infinite
8								Infinite

TABLE 1: Periodic Points for $x + y$.

(ii) Let $f(x, y) = 1 + x + y$. Then $V(f)$ contains the unit roots $(-1/2 + i\sqrt{3}/2, -1/2 - i\sqrt{3}/2)$ and $(-1/2 - i\sqrt{3}/2, -1/2 + i\sqrt{3}/2)$ so we expect that there will be infinitely many points of any period with both sides divisible by three. The set of bad periods is $L_{\langle 1+x+y \rangle} = (3\mathbb{Z})^2$. This example has positive topological entropy, the exact value of which is rather surprising. Correspondingly there are many more periodic points.

	1	2	3	4	5	6	7	8
1	3	3	9	15	33	63	129	255
2		3	9	15	33	63	129	255
3			Infinite	45	99	Infinite	387	765
4				375	825	4095	18705	57375
5					3993	21483	100749	575025
6						Infinite	815409	5081895
7							1.5 E7	8.9 E7
8								1.3 E9

TABLE 2: Periodic Points for $1 + x + y$.

(iii) Let $f(x,y) = 2 + x + y$. Then $V(f)$ contains the joint unit root $(-1, -1)$ so that there are infinitely many points of any period of the form $(2k, 2l)$ and $L_{\langle 2+x+y \rangle} = (2\mathbb{Z})^2$.

	1	2	3	4	5	6	7	8
1	4	8	28	80	244	728	2188	6560
2		Infinite	56	Infinite	488	Infinite	4376	Infinite
3			784	7280	52948	570752	4.6 E6	4.4 E7
4				Infinite	1.5 E6	Infinite	3.4 E8	Infinite
5					2.9 E7	1.8 E9	3.2 E10	1.7 E12
6						Infinite	7.6 E12	Infinite
7							3.2 E14	1.2 E17
8								Infinite

TABLE 3: Periodic Points for $2 + x + y$.

(iv) Let $f(x, y) = 3 + x + y$. This example is very different to the three above. The dynamical system is expansive; there are finitely many points of any given period. The set of bad periods is $L_{\langle 3+x+y \rangle} = \{0\}$. The entropy is easily computed and equals $\log 3$. There should be very large numbers of periodic points, reflecting the growth rate.

	1	2	3	4	5	6	7	8
1	5	15	65	255	1025	4095	16385	65535
2		45	585	3825	33825	257985	2.1 E6	1.7 E7
3			23660	606645	1.7 E7	4.8 E8	1.3 E10	3.8 E11
4				3.8 E7	3.4 E9	2.6 E11	2.1 E13	1.7 E15
5					8.7 E11	2.1 E14	5.1 E16	1.3 E19

TABLE 4: Periodic Points for $3 + x + y$.

Example C-4: We use several times results on periodic points for systems determined by non-principal ideals. These have few periodic points in the sense that their growth rate is zero, but the actual numbers involved are non-zero and depend on some elementary number theory: to find the points of given period involves inverting some matrix over a finite ring. Consider the following class of systems. Let $I(p)$ denote the ideal $\langle 1+x+y, p\rangle$ where $p \neq 0$ is a constant. By considering the problem of filling out a rectangular period $[0, m] \times [0, n]$ given a specified edge, it is clear that $|\mathrm{Fix}_{I(p)}(n, m)| \leq p^{\min\{m, n\}}$. This shows directly that the system $(X_{I(p)}, \mathbb{Z}^2)$ has zero growth rate of periodic points. If $p = 3$ we have $|\mathrm{Fix}_{I(3)}(1, n)| = |\mathrm{Fix}_{I(3)}(n, 1)| = 3$ for all $n \geq 1$. $|\mathrm{Fix}_{I(3)}(2, 2)| = 1$. If $p = 4$ then $|\mathrm{Fix}_{I(4)}(1, 1)| = 1$.

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